Homework 2 - Solutions

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1. By the definition of the absolute value, we have $|a| \geq 0$ and $|b| \geq 0$. Also, for real numbers a and b, $a \cdot 0 = b \cdot 0 = 0$. Thus, using the multiplication and transitive laws and Exercise 6 in Homework 1, we obtain $|a| \leq |b| \Rightarrow (|a|^2 \leq |a||b|$ and $|a||b| \leq |b|^2) \Rightarrow |a|^2 \leq |b|^2 \Rightarrow a^2 \leq b^2$. Hence, $|a| \leq |b| \Rightarrow a^2 \leq b^2$.

2. (a) For all rational numbers q there exists a positive real number x such that $\frac{q}{x} \geq 0$.
   (b) $x \leq 0$ and $\frac{x}{2}$ is not a natural number.
   (c) There exists an integer number y such that for all rational numbers s, $y - s \leq 0$ or $ys \neq 43$.

3. Notice that $(a-b)^2+(a-c)^2+(b-c)^2 = 2a^2+2b^2+2c^2-2ab-2ac-2bc = 2(a^2+b^2+c^2-ab-ac-bc)$. Since $a-b$, $a-c$, and $b-c$ are real numbers, $(a-b)^2 \geq 0$, $(a-c)^2 \geq 0$, and $(b-c)^2 \geq 0$. Then, $(a-b)^2+(a-c)^2+(b-c)^2 \geq 0$ what implies $2(a^2+b^2+c^2-ab-ac-bc) \geq 0$. Thus, by the multiplication law, $a^2+b^2+c^2-ab-ac-bc \geq 0$. Therefore, by the addition law, $a^2+b^2+c^2 \geq bc+ac+ab$.

4. Let x be even and y be odd. Suppose, for contradiction, that $x+y$ is even. Since x is even, $x = 2k$ for some integer k. Since $x+y$ is even, $x+y = 2m$ for some integer m. Then, $y = (x+y)-x = 2m-2k = 2(m-k)$ where $m-k$ is an integer. Thus, y is even contradicting the fact that y is odd, i.e., not even. Hence, if x is even and y is odd then $x+y$ is odd.

5. We have $x^2 - y^2 = (x-y)(x+y)$. Suppose, for contradiction, that there exist natural numbers x, y such that $(x-y)(x+y) = 1$. Since x, y are natural numbers, we have that $x+y \geq 2$, $(x-y)$ is an integer, and $x - y = \frac{1}{x+y} > 0$. In particular, $x-y \geq 1$ because $(x-y)$ is an integer and $x-y > 0$. Since $x-y \geq 1$ and $x+y \geq 2$, we have $(x-y)(x+y) \geq 2$ contradicting the fact that $(x-y)(x+y) = 1$. Therefore, there are no natural numbers x, y such that $(x-y)(x+y) = 1$.

6. We will use a proof by cases.

   Let a, b be real numbers. Using the trichotomy law, we consider three cases: both a and b are non-negative, both a and b are negative, and one of a and b is negative.

   (a) Assume $a \geq 0$ and $b \geq 0$. Then, $|a+b| = a+b = |a|+|b|$. Therefore, $|a+b| = |a|+|b|$. In particular, $|a+b| \leq |a|+|b|$.
   (b) Assume $a \leq 0$ and $b \leq 0$. Then, $a+b \leq 0$. We have $|a+b| = -(a+b) = -a+(-b) = |a|+|b|$. Therefore, $|a+b| = |a|+|b|$. In particular, $|a+b| \leq |a|+|b|$.
   (c) We now assume that one of a and b is negative. By the commutativity of operation $+$, we know that $a+b = b+a$ and $|a|+|b| = |b|+|a|$. Therefore, it does not matter which one is negative and which one is positive.

Assume $a \geq 0$ and $b < 0$. Then, $|a| = a$ and $|b| = -b$, in particular, $b = -|b|$. Then, $|a+b| = ||a| - |b||$, so $|a+b| = |a| - |b|$ if $|a| \geq |b|$ and $|a+b| = -|a| + |b|$ if $|a| \leq |b|$. Since $|a| \geq 0$ and $|b| \geq 0$, we have $-|a| \leq |a|$ and $-|b| \leq |b|$. Thus, $|a| - |b| \leq |a| + |b|$ and $-|a| + |b| \leq |a| + |b|$. Therefore, $|a+b| \leq |a| + |b|$.