**Homework 3 - Solutions**

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1. *Proof.* Let $x$ be an even integer. Then, $x = 2q$ for some integer $q$. Suppose, for the contradiction, that $x^3 - 1$ is an even integer. Then, $x^3 - 1 = 2k$ for some integer $k$. Since $x = 2q$ and $x^3 - 1 = 2k$, we have $(2q)^3 - 1 = 2k$, i.e., $8q^3 - 1 = 2k$. Thus, $1 = 8q^3 - 2k = 2(4q^3 - k)$ where $4q^3 - k$ is an integer. In particular, $4q^3 - k = \frac{1}{2}$ what implies that $\frac{1}{2}$ is an integer contradicting the fact that $\frac{1}{2}$ is not an integer. Hence, $x^3 - 1$ is not even, so $x^3 - 1$ is odd. □

2. *Proof.* The contrapositive of the statement "If $a$ is even and $b$ is odd, then $a + b$ is odd" is "If $a + b$ is even, then $a$ is odd or $b$ is even". Since the statement and its contrapositive are equivalent, it is enough to suppose $a + b$ is even and show that $a$ is odd or $b$ is even.

   If $a$ is odd, then we are done. Thus, we assume that $a$ is not odd, i.e., $a$ is even. Since $a + b$ and $a$ even, we have $a + b = 2k$ and $a = 2l$ for some integers $k$ and $l$. Then, $b = (a + b) - a = 2k - 2l = 2(k - l)$ where $(k - l)$ is an integer. Thus, $b$ is even. As a result, if $a + b$ is even, then $a$ is odd or $b$ is even. □

3. The mistake in the inductive step. The proof does not work to show that for each collection of 2 horses, all of the horses in the collection have the same color. If we remove a horse from a collection of 2 horses, then we get a collection of 1 horse, which obviously, has the same color. The problem is that the collections of 1 horse do not when we choose another horse. Therefore, we could have 2 horses such that one is of one color, and the other is of the other. Therefore, statement for 1 horse doesn't imply statement for 2 horses, so the inductive step doesn't hold.

4. *Proof.* We prove by induction.

   Base case: For $n = 1$, we have $1^2 + \ldots + n^2$ is just 1 and $\frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2\cdot1)}{6} = \frac{1\cdot2\cdot3}{6} = 1$. Therefore, for $n = 1$, we have $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   Inductive step: Suppose that for given $n \in \mathbb{N}$, we have

   $$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   Our goal is to show that

   $$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{[n+1]([n+1]+1)(2[n+1]+1)}{6},$$

   i.e.,

   $$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Adding $(n+1)^2$ to both sides of the inductive hypothesis, we get

$$\begin{aligned}1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)(n+1)}{6} \\ &= \frac{(n+1)}{6}(n(2n+1) + 6(n+1)) \\ &= \frac{(n+1)}{6}(2n^2 + 7n + 6) \\ &= \frac{(n+1)(n+2)(2n+3)}{6}\end{aligned}$$

because

$$(n+2)(2n+3) = 2n^2 + 4n + 3n + 6 = 2n^2 + 7n + 6.$$

Therefore, by the principle of mathematical induction, we proved that for each $n \in \mathbb{N}$,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

□

5. *Proof.* We prove by induction.

Base case: For $n = 1$, we have $1 + x + x^2 + \ldots + x^n = 1 + x$ and

$$\frac{x^{n+1} - 1}{x - 1} = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1} = x + 1$$

because $x - 1 \neq 0$. Therefore, for $n = 1$, the statement is true.

Inductive step: Suppose that for given $n \in \mathbb{N}$, we have

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

Our goal is to show that

$$1 + x + x^2 + \ldots + x^n + x^{n+1} = \frac{x^{[n+1]+1} - 1}{x - 1},$$

i.e.,

$$1 + x + x^2 + \ldots + x^n + x^{n+1} = \frac{x^{n+2} - 1}{x - 1}.$$

Adding $x^{n+1}$ to both sides of the inductive hypothesis, we get

$$\begin{aligned}1 + x + x^2 + \ldots + x^n + x^{n+1} &= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \\ &= \frac{x^{n+1} - 1}{x - 1} + \frac{x^{n+1}(x - 1)}{x - 1} \\ &= \frac{x^{n+1} - 1}{x - 1} + \frac{x^{n+2} - x^{n+1}}{x - 1} \\ &= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x - 1} \\ &= \frac{x^{n+2} - 1}{x - 1}.\end{aligned}$$

Therefore, by the principle of mathematical induction, we proved the statement for every $n \in \mathbb{N}$. □

6. *Proof.* We use proof by induction.

   Base case: Let $n = 1$. Then, $3^{2n-1} + 1 = 3^{2\cdot 1-1} + 1 = 4$ and 4 divides 4. Therefore, 4 divides $(3^{2n-1} + 1)$ if $n = 1$.

   Inductive step: Suppose that for given $n \in \mathbb{N}$, we have 4 divides $(3^{2n-1} + 1)$. Our goal is to show that 4 divides $(3^{2[n+1]-1} + 1)$, i.e.,4 divides $(3^{2n+1} + 1)$.

   We have

$$3^{2n+1} + 1 = 3^{2n-1} \cdot 3^2 + 1 = 3^{2n-1} \cdot (8 + 1) + 1 = (3^{2n-1} + 1) + 8 \cdot 3^{2n-1}.$$

   Since 4 divides $(3^{2n-1}+1)$ (by inductive hypothesis) and 4 divides $8 \cdot 3^{2n-1}$ (because $8 \cdot 3^{2n-1} = 4 \cdot (2 \cdot 3^{2n-1}) = 4 \cdot \textit{integer}$), we have 4 divides $(3^{2n+1} + 1)$.

   Therefore, by the principle of the mathematical induction, we proved the statement for every $n \in \mathbb{N}$. □