Homework 3

MAT 200, Instructor: Alena Erchenko

1. Prove that if x is an even integer, then $x^3 - 1$ is odd.

2. Let a and b be integer numbers. Prove using the method by contrapositive, that if a is even and b is odd, then $a + b$ is odd.

3. Find a mistake in the following argument that presumably proves that all horses have the same color.

   Proof with a mistake: It suffices to show that for each natural number n, for each collection of n horses, all of the horses in the collection have the same color. We use induction.

   Base case: Clearly, it is true for $n = 1$, because in a collection consisting of one horse, all of the horses (we have only one) in the collection have the same color.

   Inductive step: Let n be a natural number such that for each collection of n horses, all of the horses in the collection have the same color. We want to show that for each collection of $n+1$ horses, all of the horses in the collection have the same color. Removing one of the horses from the collection of $n+1$ horses, we obtain a collection of n horses, all of which have the same color by the inductive hypothesis. Removing a different horse from the collection of $n + 1$ horses, we obtain another collection of n horses, all of which have the same color by the inductive hypothesis. Hence, for each collection of $n + 1$ horses, all of the horses in the collection have the same color.

   Therefore, by the principle of mathematical induction, all horses are of the same color.

4. Prove by induction that for each natural number n,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

5. Fix a real number $x \neq 1$. Show that for every natural number n,

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

6. Prove that for all natural numbers n, we have 4 divides $(3^{2n-1} + 1)$.