**Homework 4 - Solutions**

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1. *Proof.* We prove by strong induction on $n$.

   Base case: For $n = 1$, we have $u_{m+n} = u_{m+1}$ and $u_{m-1}u_n + u_m u_{n+1} = u_{m-1}u_1 + u_m u_2 = u_{m-1} + u_m = u_{m+1}$ (by definition of Fibonacci sequence).

   For $n = 2$, we have $u_{m+n} = u_{m+2}$ and $u_{m-1}u_n + u_m u_{n+1} = u_{m-1}u_2 + u_m u_3 = u_{m-1} + 2u_m = u_{m+1} + u_m = u_{m+2}$ (by definition of Fibonacci sequence). Therefore, $u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$ for $n = 1$ and $n = 2$.

   Inductive step: Assume that for some natural number $k \geq 2$ we have $u_{m+i} = u_{m-1}u_i + u_m u_{i+1}$ for all natural numbers $2 \leq i \leq k$. We want to show that $u_{m+k+1} = u_{m-1}u_{k+1} + u_m u_{k+2}$. We have

   $$\begin{aligned} u_{m+k+1} &= u_{m+k-1} + u_{m+k} \text{ (by the definition of the sequence)} \\ &= (u_{m-1}u_{k-1} + u_m u_k) + (u_{m-1}u_k + u_m u_{k+1}) \text{ (by inductive hypothesis)} \\ &= u_{m-1}(u_{k-1} + u_k) + u_m(u_k + u_{k+1}) \\ &= u_{m-1}u_{k+1} + u_m u_{k+2} \text{ (by the definition of the sequence).} \end{aligned}$$

   As a result,$u_{m+k+1} = u_{m-1}u_{k+1} + u_m u_{k+2}$.

   By the strong principle of mathematical induction, $u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$ for all natural numbers $m \geq 2$ and $n \geq 1$. □

2. *Proof.* We have:

   $$1^3 - 2 \cdot 1^2 - 1 + 2 = 1 - 2 - 1 + 2 = 0, \text{ so } 1 \in Y;$$

   $$2^3 - 2 \cdot 2^2 - 2 + 2 = 8 - 8 - 2 + 2 = 0, \text{ so } 2 \in Y;$$

   $$3^3 - 2 \cdot 3^2 - 3 + 2 = 27 - 18 - 3 + 2 = 8 \neq 0, \text{ so } 3 \notin Y.$$

   Since $3 \in X$ and $3 \notin Y$, we have $X \not\subset Y$. □

3. (a) *Proof.* True. If $(A \setminus B) \neq \emptyset$, then $\exists x \in (A \setminus B)$. Then, $x \in A$ and $x \notin B$, so $A \not\subset B$. Therefore, $A \neq B$. □

   (b) *Proof.* False. Consider $A = \{1\}$ and $B = \{1, 2\}$. Then, $A \neq B$, $A \setminus B = \emptyset$. Notice that $B \setminus A = 2 \neq \emptyset$. □

   (c) *Proof.* False. Consider $A = \{1\}$, $B = \{2\}$, and $C = \{2\}$. Then, $C \not\subset A$.
   Also, $A \cup B = \{1, 2\}$, so $C \subset (A \cup B)$. □

4. *Proof.* First, we show that $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$. Let $x \in A \setminus (B \cup C)$. Then, $x \in A$ and $x \notin B \cup C$, i.e., $x \notin B$ and $x \notin C$. Since $x \in A$ and $x \notin B$, we have $x \in A \setminus B$. Also, since $x \in A$ and $x \notin C$, we have $x \in A \setminus C$. As a result, $x \in (A \setminus B) \cap (A \setminus C)$ because $x \in A \setminus B$ and $x \in A \setminus C$.

   Second, we show that $A \setminus (B \cup C) \supset (A \setminus B) \cap (A \setminus C)$. Let $x \in (A \setminus B) \cap (A \setminus C)$. Then, $x \in A \setminus B$ and $x \in A \setminus C$. Thus, $x \in A$ and $x \notin B$ and $x \notin C$, i.e, $x \notin B \cup C$, so $x \in A \setminus (B \cup C)$.

   Since $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cup C) \supset (A \setminus B) \cap (A \setminus C)$, we have $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. □

5. *Proof.* We prove by induction.

   Base case: Let $A$ be a set of one element, i.e., $A = \{a\}$. Then, the only subsets are $\emptyset$ and $\{a\}$. So, there are 2 subsets and $2^n = 2$ if $n = 1$.

   Inductive step: Assume that a set of $n$ elements has $2^n$ subsets. We show that a set of $n + 1$ elements has $2^{n+1}$ subsets.

   Let $A$ be a set of $n + 1$ elements, i.e., $A = \{a_1, a_2, \ldots, a_n, a_{n+1}\}$. Then, $B = \{a_1, a_2, \ldots, a_n\}$ is a set of $n$ elements and $B \subset A$, so $B$ has $2^n$ subsets by inductive hypothesis. All subsets of $A$ are either subsets of $B$ or sets that are union of $a_{n+1}$ with a subset of $B$, so there are $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ subsets of $A$. We proved inductive step.

   Therefore, by the principle of mathematical induction, we have a set of $n$ elements has $2^n$ subsets. □

6. *Proof.* Let $n \in \mathbb{N}$. Assume, for contradiction, that there is no prime number $q$ such that $n < q \leq 1 + n!$, i.e., natural numbers $n + 1, n + 2, \ldots, 1 + n!$ are not primes. Therefore, there exists a prime number $p > 1$ such that $p|(1 + n!)$. Moreover, $p \leq 1 + n!$ (by the lemma in class). Since $p$ is prime and $n + 1, n + 2, \ldots, 1 + n!$ are not primes, we have that $p \leq n$. Recall that $n! = n(n-1) \cdot \ldots \cdot 1$ and $1 < p \leq n$. Thus, $p|n!$. Since $p|n!$, $p|(1 + n!)$, and $1 = (1 + n!) - n!$, we have that $1 = pk - pl = p(k - l)$ for some integers $k, l$. Thus, $p|1$ and $p \leq 1$ contradicting the fact that $p > 1$. Therefore, there exists a prime number $q$ such that $n < q \leq 1 + n!$. □