Homework 5 - Solutions

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1. Suppose x is odd. Then, there exists $k \in \mathbb{Z}$ such that $x = 2k+1$. If $x = 2k+1$, then $x^3 = (2k+1)^3 = 2(2k^2+2k)+1$. Let $y = 2k^2+2k$. Since y is an integer and $x^3 = 2y+1$, we have x^3 is odd.

2. Let x be an odd number. Then, there exists an integer k such that $x = 2k+1$. As a result, $x^2 = (2k+1)^2 = 4(k^2+k)+1$.

   If k is an integer, then either k is even or k is odd.

   Case 1: Assume k is even, i.e., $k = 2n$ where n is an integer. Then, $k^2+k = (2n)^2+2n = 2(2n^2+n)$ and $x^2 = 8(2n^2+n)+1$. Let $y = 2n^2+n$. Then, y is integer and $x^2 = 8y+1$.

   Case 2: Assume k is odd, i.e., $k = 2n+1$ where n is an integer. Then, $k^2+k = (2n+1)^2+(2n+1) = (2n+1)(2n+2) = 2(n+1)(2n+1)$ and $x^2 = 8(n+1)(2n+1)+1$. Let $y = (n+1)(2n+1)$. Then, y is integer and $x^2 = 8y+1$.

3. First, we show that $(A \cap B) \times C \subset (A \times C) \cap (B \times C)$. Let $x \in (A \cap B) \times C$, then $x = (y, c)$, where $y \in A \cap B$ and $c \in C$. Since $y \in A \cap B$, we have $y \in A$ and $y \in B$. Therefore, $x = (y, c) \in A \times C$ and $x = (y, c) \in B \times C$, so $x \in (A \times C) \cap (B \times C)$.

   Moreover, we show that $(A \times C) \cap (B \times C) \subset (A \cap B) \times C$. Let $x \in (A \times C) \cap (B \times C)$, then $x \in (A \times C)$ and $x \in (B \times C)$. We have that $x = (y, c)$. Since $x \in (A \times C)$, we have $y \in A$ and $c \in C$. Since $x \in (B \times C)$, we have $y \in B$ and $c \in C$. Therefore, $y \in A$ and $y \in B$, so $y \in A \cap B$. As a result, $x = (y, c) \in (A \cap B) \times C$.

4. (a) We show that $\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A_n^c$.

   First, we show that $\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c \subset \bigcap_{n\in\mathbb{N}} A_n^c$. Let $x \in \left(\bigcup_{n\in\mathbb{N}} A_n\right)^c$, so $x \in A$ and $x \notin \bigcup_{n\in\mathbb{N}} A_n$, i.e., $x \notin A_n$ for all $n \in \mathbb{N}$. Therefore, $x \in A_n^c$ for all $n \in \mathbb{N}$, i.e., $x \in \bigcap_{n\in\mathbb{N}} A_n^c$.

   Moreover, we show that $\bigcap_{n\in\mathbb{N}} A_n^c \subset \left(\bigcup_{n\in\mathbb{N}} A_n\right)^c$. Let $x \in \bigcap_{n\in\mathbb{N}} A_n^c$. Then, $x \in A_n^c$ for all $n \in \mathbb{N}$, so $x \in A$ and $x \notin A_n$ for all $n \in \mathbb{N}$, i.e., $x \notin \bigcup_{n\in\mathbb{N}} A_n$. Therefore, $x \in \left(\bigcup_{n\in\mathbb{N}} A_n\right)^c$.

   (b) We show that $\left(\bigcap_{n\in\mathbb{N}} A_n\right)^c = \bigcup_{n\in\mathbb{N}} A_n^c$.

   First, we show that $\left(\bigcap_{n\in\mathbb{N}} A_n\right)^c \subset \bigcup_{n\in\mathbb{N}} A_n^c$. Let $x \in \left(\bigcap_{n\in\mathbb{N}} A_n\right)^c$, so $x \in A$ and $x \notin \bigcap_{n\in\mathbb{N}} A_n$, i.e., there exists $\tilde{n} \in \mathbb{N}$ such that $x \notin A_{\tilde{n}}$. Therefore, $x \in A_{\tilde{n}}^c$ for some $\tilde{n} \in \mathbb{N}$, i.e., $x \in \bigcup_{n\in\mathbb{N}} A_n^c$.

Moreover, we show that $\bigcup_{n\in\mathbb{N}} A_n^c \subset \left(\bigcap_{n\in\mathbb{N}} A_n\right)^c$. Let $x \in \bigcup_{n\in\mathbb{N}} A_n^c$. Then, there exists $\tilde{n} \in \mathbb{N}$ such that $x \in A_{\tilde{n}}^c$, so $x \in A$ and $x \notin A_{\tilde{n}}$ for some $\tilde{n} \in \mathbb{N}$, i.e., $x \notin \bigcap_{n\in\mathbb{N}} A_n$. Therefore, $x \in \left(\bigcap_{n\in\mathbb{N}} A_n\right)^c$.

5. We show that $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$.

   First, we show that $\{0\} \subset \bigcap_{n\in\mathbb{N}} A_n$. Notice that for any $n \in \mathbb{N}$ we have $-\frac{1}{n} < 0 < \frac{1}{n}$, so $0 \in A_n$ for any $n \in \mathbb{N}$. Therefore, $0 \in \bigcap_{n\in\mathbb{N}} A_n$, so $\{0\} \subset \bigcap_{n\in\mathbb{N}} A_n$.

   Moreover, we show that $\{0\} \supset \bigcap_{n\in\mathbb{N}} A_n$. Using the properties of the contraposition, we have that it is enough to show that if $x \notin \{0\}$, then $x \notin \bigcap_{n\in\mathbb{N}} A_n$. If $x \notin \{0\}$, then either $x > 0$ or $x < 0$.

   Case 1: If $x > 0$, then there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < x$, so $x \notin A_m$. Since $x \notin A_m$, we have that $x \notin \bigcap_{n\in\mathbb{N}} A_n$.

   Case 2: If $x < 0$, then $-x > 0$. Then, there exists $l \in \mathbb{N}$ such that $\frac{1}{l} < -x$, so $x < -\frac{1}{l}$ and $x \notin A_l$. Since $x \notin A_l$, we have that $x \notin \bigcap_{n\in\mathbb{N}} A_n$.

   As a result, we have that if $x \notin \{0\}$, then $x \notin \bigcap_{n\in\mathbb{N}} A_n$. Therefore, by the equivalence of the statement and its contraposition, we have that if $x \in \bigcap_{n\in\mathbb{N}} A_n$, then $x \in \{0\}$, i.e., $\bigcap_{n\in\mathbb{N}} A_n \subset \{0\}$.

   From the above, we have $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$.

6. We are given that $X \subset E \times F$. We need to show that $E \times F \subset \mathbb{R}^2$ and $E \times F \supset \mathbb{R}^2$.

   Let $(a, b) \in E \times F$. Then, $a \in E$ and $b \in F$. If $a \in E$, then $a \in \mathbb{R}$ because $E \subset \mathbb{R}$. If $b \in F$, then $b \in \mathbb{R}$ because $F \subset \mathbb{R}$. Therefore, $(a, b) \in \mathbb{R}^2$ because $a \in \mathbb{R}$ and $b \in \mathbb{R}$. We showed that $E \times F \subset \mathbb{R}^2$.

   Let $(a, b) \in \mathbb{R}^2$. Then, $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Therefore, by definition of X, we have $(a, a) \in X$ and $(b, b) \in X$. Since $(a, a) \in X$ and $X \subset E \times F$, we have $a \in E$. Since $(b, b) \in X$ and $X \subset E \times F$, we have $b \in E$. Therefore, $(a, b) \in E \times F$. We showed that $E \times F \supset \mathbb{R}^2$.