Homework 6

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1. Show that if x is an integer number and d is a natural number, then there exist $q \in \mathbb{Z}$ and $r \in \mathbb{N} \cup \{0\}$ such that $x = dq + r$ and $r < d$.

2. Show that there exists a unique smallest element of a non-empty collection of distinct positive integers.

   Hint: Apply Well Ordering principle for the existence. To show the uniqueness of the smallest element, assume that there are two smallest elements m_1, m_2 in the set, and, using the definition of the smallest element, show that $m_1 = m_2$.

3. In Exercise 1 you showed that if x is an integer number and d is a natural number, then there exist $q \in \mathbb{Z}$ and $r \in \mathbb{N} \cup \{0\}$ such that $x = dq + r$ and $r < d$. Show that such q and r are unique.

   Hint: Assume there exists $q_1, q_2 \in \mathbb{Z}$ and $r_1, r_2 \in \mathbb{N} \cup \{0\}$ such that $x = dq_1 + r_1$ and $x = dq_2 + r_2$ and $r_1 < d$ and $r_2 < d$. Then, show that $q_1 = q_2$ and $r_1 = r_2$.

4. (Existence of the Greatest Common Factor) Let $a, b \in \mathbb{N}$. Show that there exists $d \in \mathbb{N}$ such that

   (a) d divides a and d divides b;

   (b) If $c \in \mathbb{N}$, c divides a, and c divides b, then c divides d.

   Hint: Consider a set $X = \{ax + by | x, y \in \mathbb{Z} \quad \text{and} \quad ax + by > 0\}$. Prove that X is not empty subset of $\mathbb{Z}$ that is bounded below and apply Well Ordering Principle. As a result, you will get a smallest element (call it d) in X. The property (b) described in the definition of the greatest common divisor follows easily if you understand how d looks like since $d \in X$. To prove that d divides a, apply the division lemma to a and d, so $a = dq + r$ for some $q, r \in \mathbb{N} \cup \{0\}$ such that $r < d$. Show that $r = 0$ by contradiction, using the fact that d is the smallest in X and $r < d$. Similarly, you can show that d divides b.

5. Let u, v, and w be rational numbers. Prove the following statements using the definition of rational numbers:

   (a) $u - 2v$ is a rational number.

   (b) If $w \neq 0$, then $\frac{uv}{w}$ is a rational number.