Homework 7 - Solutions

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1. *Proof.* Let x be a rational number, i.e., there exists $p, q \in \mathbb{Z}$ such that $q \neq 0$ and $x = \frac{p}{q}$. Let y be a real number that is irrational. Then, $x + y$ is a real number. We prove that $x + y$ is irrational by contradiction. Assume $x + y$ is rational, i.e., there exists $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $x + y = \frac{a}{b}$. Then, $y = \frac{a}{b} - x = \frac{a}{b} - \frac{p}{q} = \frac{aq-bp}{bq}$. Since $y = \frac{aq-bp}{bq}$, $aq - bp$ and bq are integer numbers and $bq \neq 0$ because $b \neq 0$ and $q \neq 0$, we have y is a rational number which contradicts that we have y being irrational. Therefore, we obtain that $x + y$ is irrational. □

2. *Proof.* We have $\sqrt{3}$ is a real number. To show that $\sqrt{3}$ is an irrational number, we need to show that $\sqrt{3}$ is not a rational number. We prove it by contradiction.

   Assume that $\sqrt{3}$ is a rational. Then, by a fact in our class, we have $\sqrt{3} = \frac{a}{b}$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $GCF(a, b) = 1$. Notice that $a \in \mathbb{N}$ as $\sqrt{3} > 0$ and $b > 0$. Since $\sqrt{3} = \frac{a}{b}$, we have $3 = \left(\frac{a}{b}\right)^2$, so $3b^2 = a^2$. In particular, we have 3 divides a^2. Since 3 is a prime, we have that 3 divides a because 3 divides a^2, so $a = 3k$, where $k \in \mathbb{Z}$. Since $3b^2 = a^2$ and $a = 3k$ where $k \in \mathbb{Z}$, we obtain $3b^2 = 9k^2$, so $b^2 = 3k^2$ where $k \in \mathbb{Z}$, i.e., 3 divides b^2. Since 3 is a prime, we have that 3 divides b because 3 divides b^2. we obtained that 3 divides a and 3 divides b, so $GCF(a, b) \neq 1$. Therefore, we got a contradiction because we have $GCF(a, b) = 1$, and our assumption that $\sqrt{3}$ is rational was not right. As a result, we obtain that $\sqrt{3}$ is irrational. □

3. (a) *Proof.* We have x is rational, then by a fact in our class, we have $x = \frac{p}{q}$ where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and $GCF(p, q) = 1$. Since $x = \frac{p}{q}$, we have that $x^3+ax^2+bx+c = 0$ can be rewritten as $\left(\frac{p}{q}\right)^3 + a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0$. In particular, we obtain that $p^3 + ap^2q + bpq^2 + cq^3 = 0$, so $p^3 = q(-ap^2 - bpq - cq^2)$. Therefore, we have q divides $p \cdot p^2$ since $-ap^2 - bpq - cq^2$ is an integer. Then, there exist $q_1, q_2\mathbb{N}$ such that q_1 divides p,q_2 divides p^2, and $q = q_1q_2$. Since q_1 divides p, q_1 divides q, and $GCF(p, q) = 1$, we need to have $q_1 = 1$. Therefore, $q = q_2$, i.e., we have that q divides p^2 and $GCF(p, q) = 1$. Since q divides p^2, there exist $w_1, w_2 \in \mathbb{N}$ such that w_1 divides p, w_2 divides p, and $q = w_1w_2$. We have that w_1 divides p,w_1 divides q, and $GCF(p, q) = 1$, so $w_1 = 1$. Similarly, we have to have $w_2 = 1$. Therefore, we need to have $q = 1$, so $x = \frac{p}{q} = \frac{p}{1} = p \in \mathbb{Z}$. We obtained that if x is rational, then $x \in \mathbb{Z}$. □

   (b) *Proof.* We have that x is a real number. We prove by contradiction, that if x is not an integer, then x is irrational. We are given that x is not an integer. Assume that x is a rational number, then by the previous part of the problem, x has to be an integer, but it contradicts the fact that we are given that x is not an integer. Therefore, our assumption that x is rational is not right, and x is an irrational number. □

4. *Proof.* We prove by induction.

Base case: $n = 1$. We have a divides cb, then $\exists a_1, a_2 \in \mathbb{Z}$ such that a_1 divides c, a_2 divides b, and $a = a_1 a_2$. Moreover, we can assume that $a_2 > 0$ because if not then write $a = \tilde{a_1}\tilde{a_2}$, where $\tilde{a_1} = -a_1$ and $\tilde{a_2} = -a_2$, in particular, $\tilde{a_1}$ divides c and $\tilde{a_2}$ divides b.

As a result, a_1 divides a and a_2 divides a. Since a_2 divides b,a_2 divides a, and $GCF(a, b) = 1$, we have $a_2 = 1$ since $a_2 > 0$. Therefore, $a = a_1 \cdot 1 = a_1$ and a divides c.

Inductive step: Assume that a divides cb^n implies that a divides c. We want to prove that a divides cb^{n+1} implies that a divides c. Since $cb^{n+1} = (cb^n)b$ and a divides cb^{n+1}, we have that $\exists a_1, a_2 \in \mathbb{Z}$ such that a_1 divides cb^n, a_2 divides b, and $a = a_1 a_2$. As a above, we can assume that $a_2 > 0$. Since a_2 divides a, a_2 divides b, and $GCF(a, b) = 1$, we have $a_2 = 1$. Therefore, $a = a_1$ and a divides cb^n. By inductive hypothesis, we obtain that a divides c because a divides cb^n.

Therefore, by the principle of mathematical induction, for all $n \in \mathbb{N}$, we have that if a divides cb^n, then a divides c. □

5. *Proof.* First, we show that if 15 divides n then 3 divides n and 5 divides n.

   If 15 divides n, then $n = 15k$ for some integer k. In particular, $n = 3 \cdot (5k) = 5 \cdot (3k)$, where $(5k), (3k) \in \mathbb{Z}$. Thus, 3 divides n and 5 divides n.

   Second, we show that if 3 divides n and 5 divides n, then 15 divides n.

   If 3 divides n, then $n = 3m$ for some $m \in \mathbb{Z}$. Therefore, in that case if 5 divides n, then 5 divides $3m$. Since 5 is a prime, then 5 divides m and $m = 5l$ for some $l \in \mathbb{Z}$ because 5 does not divide 3. As a result, $n = 3 \cdot (5l) = 15l$ where $l \in \mathbb{Z}$. Thus, 15 divides n. □

6. *Solution.* We have

$$136 = 2^3 \cdot 17, \qquad 150 = 2 \cdot 3 \cdot 5^2, \qquad 255 = 3 \cdot 5 \cdot 17, \qquad 1980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11.$$

   Therefore,

$$GCF(136, 150) = 2$$

   and

$$GCF(255, 1980) = 3 \cdot 5 = 15.$$

□