Homework 9 - Solutions

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1. *Solution.* We show that there is no $x \in \mathbb{Z}$ such that $15x \equiv 5 \pmod 3$. Assume, for contradiction, that there exists $x \in \mathbb{Z}$ such that $15x \equiv 5 \pmod 3$. Then, there exists $k \in \mathbb{Z}$ such that $5 - 15x = 3k$, so $5 = 3(k - 5x)$ where $(k - 5x) \in \mathbb{Z}$. Thus, 3 divides 5 contradicting the fact that 3 doesn't divide 5. □

2. *Solution.* To determine whether f is an injection:

   Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$, i.e., $(3x_1^2, x_1^2 - 3) = (3x_2^2, x_2^2 - 3)$. Then, $3x_1^2 = 3x_2^2$ and $x_1^2 - 3 = x_2^2 - 3$. Then, $x_1^2 = x_2^2$. So, $x_1 = x_2$ or $x_1 = -x_2$. Since $Dom(f) = \mathbb{R}$, both options are possible. Notice that $f(1) = (3, -2)$ and $f(-1) = (3, -2)$. So, $f(1) = f(-1)$. Therefore, f is not an injection.

   To determine whether f is a surjection:

   We have $Dom(f) = \mathbb{R}$ and $Codom(f) = \mathbb{R}^2$. Consider $(a, b) \in Codom(f)$, i.e., $a \in \mathbb{R}$ and $b \in \mathbb{R}$. We want to see if there exists $x \in \mathbb{R}$, such that $f(x) = (a, b)$. Assume such x exists, then $f(x) = (a, b)$, so $(3x^2, x^2 - 3) = (a, b)$. Then, $3x^2 = a$ and $x^2 - 3 = b$. In particular, to have solution for $3x^2 = a$, we need to have $a \geq 0$. Notice that $(-1, 2) \in \mathbb{R}^2$, and there is no $x \in \mathbb{R}$ such that $f(x) = (-1, 2)$ because $-1 < 0$. Therefore, f is not a surjection. □

3. *Solution.* To determine whether f is an injection:

   Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$, i.e., $(3x_1, x_1^2 - 3) = (3x_2, x_2^2 - 3)$. Then, $3x_1 = 3x_2$ and $x_1^2 - 3 = x_2^2 - 3$. Then, $x_1 = x_2$ and $x_1^2 = x_2^2$. So, $x_1 = x_2$. Therefore, f is an injection.

   To determine whether f is a surjection:

   We have $Dom(f) = \mathbb{R}$ and $Codom(f) = \mathbb{R}^2$. Consider $(a, b) \in Codom(f)$, i.e., $a \in \mathbb{R}$ and $b \in \mathbb{R}$. We want to see if there exists $x \in \mathbb{R}$, such that $f(x) = (a, b)$. Assume such x exists, then $f(x) = (a, b)$, i.e., $(3x, x^2 - 3) = (a, b)$. Then, $3x = a$ and $x^2 - 3 = b$. In particular, $x = \frac{a}{3}$ and $b = \left(\frac{a}{3}\right)^2 - 3$. Notice that $(3, 2) \in \mathbb{R}^2$, i.e., $(3, 2) \in Codom(f)$. If there exists $x \in \mathbb{R}$ such that $f(x) = (3, 2)$, then, by the above, we have $x = 1$ and $2 = 1^2 - 3 = -2$. Since $2 \neq -2$, there is no $x \in \mathbb{R}$ such that $f(x) = (3, 2)$. Therefore, f is not a surjection. □

4. *Proof.* We have that if $x \in A \cup B$, then $h(x) = f(x)$ if $x \in A$ or $h(x) = g(x)$ if $x \in B$. Therefore, if $x \in A \cap B$, then $h(x) = f(x)$ and $h(x) = g(x)$. Therefore, to have h to be well-defined, we should have $f(x) = g(x)$ for all $x \in A \cap B$. □

5. *Proof.* Notice that $Dom(f) = \mathbb{R}^2$ and $Codom(f) = \mathbb{R}^2$. To show that f is a bijection, we need to show that f is an injection and a surjection.

   Injection: Assume that $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $f(x_1, y_1) = f(x_2, y_2)$. Then, $(x_1 - y_1, 2x_1) = (x_2 - y_2, 2x_2)$, so $x_1 - y_1 = x_2 - y_2$ and $2x_1 = 2x_2$. Since $2x_1 = 2x_2$, we have that $x_1 = x_2$, so

$x_1 - y_1 = x_2 - y_1$. Therefore, $x_2 - y_1 = x_2 - y_2$ because $x_1 - y_1 = x_2 - y_2$ and $x_1 - y_1 = x_2 - y_1$. Then, $y_1 = y_2$. As a result, we have $(x_1, y_1) = (x_2, y_2)$ because $x_1 = x_2$ and $y_1 = y_2$. So, f is injective.

Surjection: Let $(a, b) \in \mathbb{R}^2$. We want to find $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = (a, b)$. If $f(x, y) = (a, b)$, then $(x-y, 2x) = (a, b)$, i.e., $x-y = a$ and $2x = b$, so $x = \frac{b}{2}$ and $y = x-a = \frac{b}{2}-a$. Notice that $\frac{b}{2} \in \mathbb{R}$ and $\left(\frac{b}{2} - a\right) \in \mathbb{R}$. Therefore, for any $(a, b) \in \mathbb{R}^2$ there exists $(x, y) = (\frac{b}{2}, \frac{b}{2} - a) \in \mathbb{R}^2$ such that $f(x, y) = f(\frac{b}{2}, \frac{b}{2} - a) = (a, b)$. So, f is surjective.

Since f is injective and surjective, we have that f is bijective. □

6. *Proof.* Let $X = \{1\}$ and $Y = \{-1, 1\}$. Define $f(1) = 1$ and $g(y) = y^2$ if $y \in Y$. Notice that $\{1\} \subset Y$, so $f\colon X \to Y$. Moreover, $(-1)^2 = 1^2 = 1$, so $g\colon Y \to X$. Then, f is not surjective because $-1 \in Y$ but $-1 \notin Im(f)$, g is not injective because $-1 \neq 1$, $-1, 1 \in Y$ and $g(-1) = g(1)$. We have $g \circ f(1) = g(f(1)) = g(1) = 1^1 = 1$. Since $X = \{1\}$ and $g \circ f(1) = 1$, we have that $g \circ f = id_X$. □