Homework 10 - Solutions

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1. (a) *Proof.* Since B is non-empty, then $\exists b \in B$. We have $Codom(\pi_A) = A$. Let $a \in A$. Then, $(a, b) \in A \times B$ because $a \in A$ and $b \in B$. Moreover, $\pi_A(a, b) = a$. Therefore, by definition of surjection, π_A is a surjection because a was any element in A. □

   (b) *Proof.* If B has only one element, then π_A is an injection. Let $B = \{b\}$. Then, $A \times B = \{(x, b) | x \in A\}$. Assume $(a_1, b_1), (a_2, b_2) \in A \times B$ and $\pi_A(a_1, b_1) = \pi_A(a_2, b_2)$. Since $(a_1, b_1), (a_2, b_2) \in A \times B$, then $b_1 = b_2 = b$ and $a_1, a_2 \in A$. Moreover, $\pi_A(a_1, b_1) = a_1$ and $\pi_A(a_2, b_2) = a_2$, so $a_1 = a_2$. Therefore, $(a_1, b_1) = (a_2, b_2)$, so π_A is an injection if B has only one element.

   If b has at least two distinct elements, then pi_A is not an injection. Because let $b_1, b_2 \in B$ and $b_1 \neq b_2$. Let $a \in A$. Then, $\pi_A(a, b_1) = a = \pi_A(a, b_2)$, but $(a, b_1) \neq (a, b_2)$. Therefore, π_A is not an injection in that case. □

2. *Proof.* $\Leftarrow$: Assume there exists a function $g\colon Y \to X$ such that $f \circ g = id_Y$. We show that f is surjective. Notice that since $f \circ g = id_Y$ and id_Y is a well-defined function on Y, we have that $f \circ g$ is a well-defined function on Y. Let $y \in Y$. Then, $(f \circ g)(y) = f(g(y)) = id_Y(y) = y$. Moreover, $g(y) \in X$ because g is a function and $Codom(g) = X$. In particular, $g(y) \in Dom(f)$ because $Dom(f) = X$. As a result, for any $y \in Codom(f)$ there exists $x_y = g(y) \in Dom(f)$ such that $f(x_y) = y$. Therefore, f is surjective.

   $\Rightarrow$: Assume f is surjective. We show that there exists a function $g\colon Y \to X$ such that $f \circ g = id_Y$. Since f is surjective, for any $y \in Y$ there exists $x \in X$ such that $f(x) = y$. For any $y \in Y$ choose $x_y \in X$ such that $f(x_y) = y$ and define $g(y) = x_y$. Then, g is a well defined function from Y to X because it is defined for all $y \in Y$ and any $y \in Y$ is mapped to only one point in X. In particular, $Im(g) \subset Dom(f)$ and $f \circ g$ is a well-defined function from Y to Y. Moreover, for any $y \in Y$, we have that $(f \circ g)(y) = f(g(y)) = f(x_y) = y$. So, $f \circ g = id_Y$. Therefore, we constructed a function g with desired properties. □

3. *Proof.* $\Leftarrow$: Assume there exists a function $g\colon Y \to X$ such that $g \circ f = id_X$. We show that f is injective. Notice that since $g \circ f = id_X$ and id_X is a well-defined function on X, we have that $g \circ f$ is a well-defined function on X. Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Then,$g(f(x_1)) = g(f(x_2))$ because g is a function on Y and $Codom(f) = Y$. Moreover, we have $x_1 = id_X(x_1) = (g \circ f)(x_1) = g(f(x_1))$ and $x_2 = id_X(x_2) = (g \circ f)(x_2) = g(f(x_2))$ because $g \circ f = id_X$. As a result, we obtain that $x_1 = x_2$. So, f is injective.

   $\Rightarrow$: Assume f is injective. We show that there exists a function $g\colon Y \to X$ such that $g \circ f = id_X$. Since f is injective, so for any $y \in Im(f)$ there exists a unique $x_y \in Dom(f)$ such that $f(x_y) = y$. Define $g(y) = x_y$ for any $y \in Im(f)$. Recall that X is non-empty, so choose some $a \in X$. Define $g(y) = a$ for any $y \in Codom(f) \setminus Im(f)$. Then, g is a well defined function from Y to X because it is defined for all $y \in Y$ and any $y \in Y$ is mapped to only one point in X.

Also, $Im(f) \subset Dom(g)$ and $g \circ f$ is a well-defined function from X to X. For any $x \in X$ there exists $y \in Im(f)$ such that $x = x_y$, so

$$(g \circ f)(x) = (g \circ f)(x_y) = g(f(x_y)) = g(y) = x_y = x.$$

Therefore, $g \circ f = id_X$. □

4. *Proof.* We claim that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(y) = y^{\frac{1}{3}} + 1$ for all $y \in \mathbb{R}$ is the inverse of f.

   Notice that g is well-defined on $\mathbb{R}$.

   For all $x \in \mathbb{R}$, we have

$$g \circ f(x) = g(f(x)) = g((x-1)^3) = \left((x-1)^3\right)^{\frac{1}{3}} + 1 = x - 1 + 1 = x.$$

   Thus, $g \circ f = id_{\mathbb{R}}$.

   Moreover, for all $y \in \mathbb{R}$, we have

$$f \circ g(y) = f(g(y)) = f(y^{\frac{1}{3}} + 1) = ((y^{\frac{1}{3}} + 1) - 1)^3 = \left(y^{\frac{1}{3}}\right)^3 = y.$$

   Thus, $f \circ g = id_{\mathbb{R}}$.

   By the definition of the inverse of a function, we have g is the inverse of f, i.e., $f^{-1} = g$. □

5. *Proof.* Since A has n elements, there exists a bijection $f\colon A \to \{1, 2, \ldots, n\}$. Since B has m elements, there exists a bijection $g\colon B \to \{1, 2, \ldots, m\}$.

   Define $\tilde{g}\colon B \to \{n{+}1, n{+}2, \ldots, n{+}m\}$ by $\tilde{g}(b) = n{+}g(b)$. Notice that $\tilde{g}$ is a function because g is a function and to each value in B we assigned only one value in $\{n{+}1, n{+}2, \ldots, n{+}m\}$. We show that $\tilde{g}$ is a bijection. Assume $b_1, b_2 \in B$ such that $\tilde{g}(b_1) = \tilde{g}(b_2)$. Then, $n + g(b_1) = n + g(b_2)$, so $g(b_1) = g(b_2)$ what implies that $b_1 = b_2$ because g is an injection. Therefore, $\tilde{g}$ is an injection. For any $k \in \{n+1, n+2, \ldots, n+m\}$ we have $(k - n) \in \{1, 2, \ldots, m\}$. Then, $\tilde{g}(g^{-1}(k-n)) = n + g(g^{-1}(k-n)) = n + k - n = k$ where $g^{-1}\colon \{1, 2, \ldots m\} \to B$ is the inverse of g which exists because g is a bijection. Therefore, $\tilde{g}$ is a surjection.

   Define $h\colon A \cup B \to \{1, 2, \ldots, n, n+1, n+2 \ldots, n+m\}$ as

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \tilde{g}(x) & \text{if } x \in B. \end{cases}$$

   Notice that by definition h is a function because $A \cap B = \emptyset$, so we assigned only one value for each $x \in A \cup B$. Moreover, $Im(h) = Im(f) \cup Im(\tilde{g}) = \{1, 2, \ldots, n, n+1, n+2 \ldots, n+m\}$ because f and $\tilde{g}$ are surjective on A and B, respectively. Therefore, h is surjective. Also, $Im(f) \cap Im(\tilde{g}) = \emptyset$ by the definition, f and $\tilde{g}$ are injective, so h is injective. Thus, h is a bijection because h is injective and surjective. Therefore, $A \cup B$ has $n + m$ elements. □

6. *Proof.* We prove by contradiction. Assume $A \setminus \{a_1, a_2, \ldots, a_n\}$ is finite, then there exist $m \in \mathbb{N} \cup \{0\}$ and a bijection $f\colon A \setminus \{a_1, a_2, \ldots, a_n\} \to \{1, 2, \ldots, m\}$.

   Define a function $g\colon \{a_1, a_2, \ldots, a_n\} \to \{m+1, m+2, \ldots, m+n\}$ as $g(a_k) = m + k$ for $k \in \{1, 2, \ldots, n\}$. Since $a_i \neq a_j$ if $i \neq j$, we have a well-defined function. Moreover, g is surjective because for any $l \in \{m+1, m+2, \ldots, m+n\}$ we have $(l-m) \in \{1, 2, \ldots, n\}$ and $g(a_{l-m}) = m+l-m = l$. Also, it is injective because if $a_i, a_j \in \{a_1, a_2, \ldots, a_n\}$ and $g(a_i) = g(a_j)$ then $m+i = m+j$,so $i = j$, so $a_i = a_j$.Thus, g is a bijection.

Notice that $A = (A \setminus \{a_1, a_2, \ldots, a_n\}) \cup \{a_1, a_2, \ldots, a_n\}$. Define $h\colon A \to \{1, 2, \ldots, m, m+1, m+2, \ldots, m+n\}$ in the following way:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \setminus \{a_1, a_2, \ldots, a_n\}, \\ \tilde{g}(x) & \text{if } x \in \{a_1, a_2, \ldots, a_n\}. \end{cases}$$

Notice that by definition h is a function because $(A \setminus \{a_1, a_2, \ldots, a_n\}) \cap \{a_1, a_2, \ldots, a_n\} = \emptyset$, so we assigned only one value for each $x \in A$. Moreover, $Im(h) = Im(f) \cup Im(g) = \{1, 2, \ldots, m, m+1, m+2 \ldots, m+n\}$ because f and g are surjective on $A \setminus \{a_1, a_2, \ldots, a_n\}$ and $\{a_1, a_2, \ldots, a_n\}$, respectively. Therefore, h is surjective. Also, $Im(f) \cap Im(g) = \emptyset$ by the definition, f and g are injective, so h is injective. Thus, h is a bijection because h is injective and surjective. Therefore, A has $n+m$ elements, so A is finite. Since A is infinite, we obtained a contradiction. Therefore, $A \setminus \{a_1, a_2, \ldots, a_n\}$ is an infinite set. $\square$