Homework 10

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1. Let A and B be non-empty sets. Consider a projection function $\pi_A : A \times B \to A$ defined by $\pi_A(a, b) = a$ for all $(a, b) \in A \times B$.
   (a) Prove that π_A is a surjection.
   (b) Is π_A injective? (Hint: The answer depends on how many elements B has.)
2. Let X, Y be non-empty sets. Let $f \colon X \to Y$ be a function. Show that f is surjective if and only if there is a function $g \colon Y \to X$ such that $f \circ g = id_Y$.
3. Let X, Y be non-empty sets. Let $f \colon X \to Y$ be a function. Show that f is injective if and only if there is a function $g \colon Y \to X$ such that $g \circ f = id_X$.
4. Find the inverse for the function $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x-1)^3$ for all $x \in \mathbb{R}$. Explain your answer.
5. Let $n, m \in \mathbb{N}$. Let A be a set with n elements and B be a set with m elements. Show that if $A \cap B = \emptyset$, then $A \cup B$ has $n + m$ elements.

   Hint: Remember that by definition a set X has n elements if X has the same cardinality as $\{1, 2, \ldots, n\}$. Therefore, to prove the above result you need to construct a bijection between $A \cup B$ and $\{1, 2, \ldots, n + m\}$.
6. Let A be an infinite set. Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A$ such that $a_i \neq a_j$ if $i \neq j$. Prove that $A \setminus \{a_1, a_2, \ldots, a_n\}$ is an infinite set.