Homework 11 - Solutions

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1. (a) *Proof.* We need to assign only one value from $\{1, 2, \ldots, n\}$ to each element of $\{1, 2, \ldots, m\}$. The set $\{1, 2, \ldots, m\}$ has m elements, for each element there exist n options of what to assign. Thus, there are n^m distinct functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. $\square$

   (b) *Proof.* If $n < m$, then there are no injective functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$ because $\{1, 2, \ldots, m\}$ has m elements and to obtain injective function we need to assign different values to different elements of $\{1, 2, \ldots, m\}$, so we need to have at least m different elements in $\{1, 2, \ldots, n\}$ which has only n elements.

   If $n \geq m$, then there are $n(n-1) \cdot \ldots \cdot (n-m+1) = \frac{n!}{(n-m)!}$ distinct injective functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. The set $\{1, 2, \ldots, m\}$ has m elements. Let's first assign value of a function to 1, we have n options. Then, we assign value of the function to 2, which can be any element in $\{1, 2, \ldots, n\}$ except the element that was assigned to 1 because we want an injective function, so we have $n-1$ options. Then, we assign value of the function to 3, which can be any element in $\{1, 2, \ldots, n\}$ except the elements that were assigned to 1 and 2 because we want an injective function, so we have $n-2$ options. And so on. $\square$

2. First, notice that $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$ because $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B \Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \notin A$ and $x \in B)$ or $(x \in A$ and $x \in B) \Leftrightarrow x \in A \setminus B$ or $x \in B \setminus A$ or $x \in A \cap B \Leftrightarrow x \in (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$. Moreover, $A \setminus B) \cap (B \setminus A) = \emptyset$, $(A \setminus B) \cap (A \cap B) = \emptyset$, and $(B \setminus A) \cap (A \cap B) = \emptyset$. In particular, $(A \setminus B) \cap (B \setminus A)) \cap (A \cap B) = \emptyset$. Moreover, $A \setminus B \subset A$, $B \setminus A \subset B$, and $A \cap B \subset A$, thus, $A \setminus B$, $B \setminus A$, and $A \cap B$ are finite since A and B are finite. Using Problem 5 in Homework 10 twice, we obtain

$$\begin{aligned} |A \cup B| &= |(A \setminus B) \cup (B \setminus A) \cup (A \cap B)| \\ &= |(A \setminus B) \cup (B \setminus A)| + |A \cap B| \\ &= |A \setminus B| + |B \setminus A| + |A \cap B|. \end{aligned}$$

Also, $A = (A \setminus B) \cup (A \cap B)$ because $x \in A \Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \in B) \Leftrightarrow x \in (A \setminus B)$ or $x \in (A \cap B) \Leftrightarrow x \in (A \setminus B) \cup (A \cap B)$. Notice that $(A \setminus B) \cap (A \cap B) = \emptyset$. Applying Problem 5 in Homework 10, we obtain

$$|A| = |(A \setminus B) \cup (A \cap B)| = |A \setminus B| + |A \cap B|,$$

so

$$|A \setminus B| = |A| - |A \cap B|.$$

Similarly,

$$|B \setminus A| = |B| - |A \cap B|.$$

Combining all equalities together, we obtain

$$\begin{aligned}|A \cup B| &= |A \setminus B| + |B \setminus A| + |A \cap B| \\ &= |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B| \\ &= |A| + |B| - |A \cap B|.\end{aligned}$$

3. *Proof.* We showed in class that there exists a bijection $f\colon \mathbb{Z} \to \mathbb{N}$. Let $g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{N} \times \mathbb{N}$ be defined as $g(a,b) = (f(a), f(b))$ for any $a, b \in \mathbb{Z}$. We show that g is a bijection. Assume $(a_1, b_1), (a_2, b_2) \in \mathbb{Z} \times \mathbb{Z}$ such that $g(a_1, b_1) = g(a_2, b_2)$. Then, $(f(a_1), f(b_1)) = (f(a_2), f(b_2))$, so $f(a_1) = f(a_2)$ and $f(b_1) = f(b_2)$. Therefore, $a_1 = a_2$ and $b_1 = b_2$ because f is a bijection (in particular, injection), so $(a_1, b_1) = (a_2, b_2)$. Thus, g is an injection. For any $(n, m) \in \mathbb{N} \times \mathbb{N}$ we have $(f^{-1}(n), f^{-1}(m)) \in \mathbb{Z} \times \mathbb{Z}$ where f^{-1} is the inverse of f which exists because f is a bijection. Then, $g((f^{-1}(n), f^{-1}(m))) = (f(f^{-1}(n)), f(f^{-1}(m))) = (n, m)$. Thus, g is a surjection. Therefore, g is a bijection because it is a surjection and an injection.

   Also, in class we showed that there exists a bijection $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. As a result, by theorem in class about composition of bijections, we have that $h \circ g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ is a bijection, so $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$. Therefore, $\mathbb{Z} \times \mathbb{Z}$ is countable. □

4. *Proof.* Let a be a repeating decimal. Then, $a = 0.y_1 y_2 \ldots y_n \overline{x_1 x_2 \ldots x_k}$ where $y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_k \in \{0, 1, 2, \ldots, 9\}$, $n \in \mathbb{N} \cup \{0\}$, and $k \in \mathbb{N}$. We have

$$a \cdot 10^n = y_1 y_2 \ldots y_n.\overline{x_1 x_2 \ldots x_k}$$

   and

$$a \cdot 10^{n+k} = y_1 y_2 \ldots y_n x_1 x_2 \ldots x_k.\overline{x_1 x_2 \ldots x_k}.$$

   Then,

$$a(10^{n+k} - 10^n) = a \cdot 10^{n+k} - a \cdot 10^n = y_1 y_2 \ldots y_n x_1 x_2 \ldots x_k - y_1 y_2 \ldots y_n.$$

   Since $(y_1 y_2 \ldots y_n x_1 x_2 \ldots x_k - y_1 y_2 \ldots y_n) \in \mathbb{Z}$ and $(10^{n+k} - 10^n) = 10^n(10^k - 1) \in \mathbb{N}$ because $k \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have that $a = \frac{y_1 y_2 \ldots y_n x_1 x_2 \ldots x_k - y_1 y_2 \ldots y_n}{10^{n+k} - 10^n}$ is a rational number. □

5. *Proof.* If $x \in A$, then $\{x\} \in \mathcal{P}(A)$. Define $h\colon A \to \mathcal{P}(A)$ by setting $h(x) = \{x\}$. Then, h is a well-defined function as to each element of A we prescribed only one element of $\mathcal{P}(A)$.

   Assume $x, y \in A$ such that $h(x) = h(y)$. Then, $\{x\} = \{y\}$, so $x \in \{y\}$ what implies $x = y$. Therefore, h is an injection.

   Since $h\colon A \to \mathcal{P}(A)$ is an injection, we have that $|A| \le |\mathcal{P}(A)|$. □

6. *Proof.* A polynomial in x of degree $n \in \mathbb{N}$ with rational coefficients has form $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ where $a_0 \in \mathbb{Q} \setminus \{0\}$ and $a_1, a_2, \ldots, a_n \in \mathbb{Q}$. Moreover, two polynomials in x are the same if and only if they have the same coefficients. Thus, we can code each polynomial in x of degree $n \in \mathbb{N}$ with rational coefficients by a sequence $(a_0, a_1, a_2, \ldots, a_{n-1}, a_n)$ where $a_0 \in \mathbb{Q} \setminus \{0\}$ and $a_1, a_2, \ldots, a_n \in \mathbb{Q}$. Thus, the problem can be formulated to show that the set $P_n = \{(a_0, a_1, a_2, \ldots, a_{n-1}, a_n) | a_0 \in \mathbb{Q} \setminus \{0\}, a_1, a_2, \ldots, a_n \in \mathbb{Q}\}$ is countable for all $n \in \mathbb{N}$.

   We prove the statement by induction on n.

Base case: Let $n = 1$. Then, $P_n = P_1 = \{(a_0, a_1)|a_0 \in \mathbb{Q} \setminus \{0\}, a_1 \in \mathbb{Q}\} = (\mathbb{Q} \setminus \{0\}) \times \mathbb{Q}$. Recall by the example in class, we showed that $\mathbb{Q}$ is countable, so there exists a bijection $f\colon \mathbb{Q} \to \mathbb{N}$. Also, by the fact that we should in class, $\mathbb{Q} \setminus \{0\}$ is countable because it is infinite (by Problem 6 in Homework 10 as $\mathbb{Q}$ is infinite) and a subset of a countable set (as $\mathbb{Q} \setminus \{0\} \subset \mathbb{Q}$ and $\mathbb{Q}$ is countable). Thus, there exists a bijection $g\colon \mathbb{Q} \setminus \{0\} \to \mathbb{N}$.

Let $h\colon (\mathbb{Q} \setminus \{0\}) \times \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$ be defined by $h(x, y) = (g(x), f(y))$ for all $(x, y) \in (\mathbb{Q} \setminus \{0\}) \times \mathbb{Q}$. Consider a function $y\colon \mathbb{N} \times \mathbb{N} \to (\mathbb{Q} \setminus \{0\} \times \mathbb{Q})$ defined by $y(n, m) = (g^{-1}(n), f^{-1}(m))$. Then, y is the inverse of h because $y(h(x, y)) = y(g(x), f(y)) = (g^{-1}(g(x)), f^{-1}(f(y))) = (x, y)$ for all $(x, y) \in (\mathbb{Q} \setminus \{0\}) \times \mathbb{Q}$ and $h(y(n, m)) = h(g^{-1}(n), f^{-1}(m)) = (g(g^{-1}(n)), f(f^{-1}(m))) = (n, m)$ for all $(n, m) \in \mathbb{N} \times \mathbb{N}$. Thus, h is a bijection so $|(\mathbb{Q} \setminus \{0\} \times \mathbb{Q})| = |\mathbb{N} \times \mathbb{N}|$. Using the fact that $\mathbb{N} \times \mathbb{N}$ is countable and the composition of bijections is a bijection, we obtain $(\mathbb{Q} \setminus \{0\} \times \mathbb{Q})$ is countable so P_1 is countable.

Inductive step: Assume for some $k \in \mathbb{N}$ we have that P_k is countable. We want to show that P_{k+1} is countable. Define a function $f_1\colon P_{k+1} \to P_k \times \mathbb{Q}$ defined by

$$f_1((a_0, a_1, \ldots, a_k, a_{k+1})) = ((a_0, a_1, \ldots, a_k), a_{k+1})\,.$$

for all $(a_0, a_1, \ldots, a_k, a_{k+1}) \in P_{k+1}$. We have that f_1 is a bijection because it has an inverse $g_1\colon P_k \times \mathbb{Q} \to P_{k+1}$ defined by $g_1(((b_0, b_1, \ldots, b_k), b)) = (b_0, b_1, b_2, \ldots, b_k, b)$. Therefore,

$$|P_{k+1}| = |P_k \times \mathbb{Q}|.$$

Since P_k is countable, there exists a bijection $f_2\colon P_k \to \mathbb{N}$. Let $f\colon \mathbb{Q} \to \mathbb{N}$ be a bijection which exists because $\mathbb{Q}$ is countable. Then, $h_1\colon P_k \times \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$ defined by $h_1(((b_0, b_1, \ldots, b_k), b)) = (f_2((b_0, b_1, \ldots, b_k)), f(b))$ for all $((b_0, b_1, \ldots, b_k), b) \in P_k \times \mathbb{Q}$ is a bijection (the proof is similar as in the base case). Thus, $|P_k \times \mathbb{Q}| = |\mathbb{N} \times \mathbb{N}|$. Using the fact that composition of bijections is a bijection and $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, we obtain $|P_{k+1}| = |\mathbb{N}|$ so P_{k+1} is countable.

As a result, by the principle of mathematical induction, P_n is countable for all $n \in \mathbb{N}$ so the set of polynomials in x of degree n with rational coefficients is countable. □

7. *Proof.* By Problem 5, we have that $|\mathbb{N}| \leq |\mathcal{P}(\mathbb{N})|$, so $\mathcal{P}(\mathbb{N})$ is infinite. Therefore, we need to show that $\mathcal{P}(\mathbb{N})$ is not countable. We prove that statement by contradiction. Assume $\mathcal{P}(\mathbb{N})$ is countable, then there exists a bijection $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$. We assign to each $A \in \mathcal{P}(\mathbb{N})$ an infinite sequence $(y_1, y_2, y_3, \ldots)$, where $y_i = 1$ if $i \in A$ and $y_i = 0$ if $i \notin A$. Notice that different subsets of $\mathbb{N}$ have different sequences because if $A, B \in \mathcal{P}(\mathbb{N})$ and $A \neq B$, then there exists $k \in \mathbb{N}$ such that $[k \in A$ and $k \notin B]$ or $[k \notin A$ and $k \in B]$, so k-th element of the sequence for A is different from the k-th element of the sequence for B. Moreover, any sequence $(x_1, x_2, \ldots)$ where $x_i \in \{0, 1\}$ for all $i \in \mathbb{N}$ corresponds to a set $A = \{i \in \mathbb{N}|x_i = 1\}$, so $A \in \mathcal{P}(\mathbb{N})$.

   For any $n \in \mathbb{N}$, we have $f(n) \in \mathcal{P}(\mathbb{N})$, so there exists a sequence $(y_1^n, y_2^n, \ldots)$ coding $f(n)$. Consider a sequence $(x_1, x_2, \ldots)$ where $x_i = 1$ if $y_i^i = 0$ and $x_i = 0$ if $y_i^i = 1$ for all $i \in \mathbb{N}$. Let $A = \{i \in \mathbb{N}|x_i = 1\}$. Then, for any $n \in \mathbb{N}$ we have that $f(n) \neq A$ because the sequences are not equal. So, $A \in \mathcal{P}(\mathbb{N})$ and $A \notin Im(f)$. Therefore, f is not surjective, so f is not bijective. We obtained a contradiction. Therefore, $\mathcal{P}(\mathbb{N})$ is not countable.

   Thus, $\mathcal{P}(\mathbb{N})$ is uncountable. □