Homework 11

MAT 200, Instructor: Alena Erchenko

1. Let $n, m \in \mathbb{N}$.

   (a) How many distinct functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$ are there? Explain your answer.

   (b) How many distinct injective functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$ are there? Explain your answer.

2. Suppose that A and B are finite sets. Prove that $A \cup B$ is a finite set and $|A \cup B| = |A| + |B| - |A \cap B|$. Hint: Show that $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$, $A = (A \setminus B) \cup (A \cap B)$, and use Problem 5 in Homework 10.

3. Prove that $\mathbb{Z} \times \mathbb{Z}$ is countable.

4. Prove that a repeating decimal is a rational number.

   Hint: Recall that a repeating decimal a can be written as $a = 0.y_1 y_2 \ldots y_n \overline{x_1 x_2 \ldots x_k}$ where $y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_k \in \{0, 1, 2, \ldots, 9\}$ and $x_1 x_2 \ldots x_k$ is repeated indefinitely, $n \in \mathbb{N} \cup \{0\}$, and $k \in \mathbb{N}$.

5. Let A be a set. Let $\mathcal{P}(A) = \{C | C \subset A\} = \{\text{all subsets of } A\}$. Show that $|A| \leq |\mathcal{P}(A)|$.

6. Let $n \in \mathbb{N}$. Prove that the number of polynomials in x of degree n with rational coefficients is countable.

7. Using the idea from the proof that $\mathbb{R}$ is uncountable, show that $\mathcal{P}(\mathbb{N})$ is uncountable. Do not use Cantor's generalized diagonal lemma or its proof.

   Hint: Think how to code subsets in terms of a sequence of zeros and ones.