Homework 1

MAT 351, Instructor: Alena Erchenko

Everywhere below [HK] stands for the book "A First Course in Dynamics with a panorama of recent developments" by B.Hasselblatt and A.Katok.

1. (Exercise 1.2.4 in [HK]) Consider the sequence of numbers $\{b_i\}_{i=1}^{\infty}$ such that $b_0 = b_1 = 1$ and $b_{n+1} = b_n + b_{n-1}$ for $n \in \mathbb{N}$. For all $n \in \mathbb{N} \cup \{0\}$, express $\sum_{i=0}^{n} b_i$ in terms of b_{n+2}.

2. (Exercise 1.2.18 in [HK]) Denote by a_n the number of sequences of 0's and 1's of length n that do not have two consecutive 0's. Show that $a_1 = 2, a_2 = 3$ and $a_{n+1} = a_n + a_{n-1}$ for $n \in \mathbb{N} \setminus \{1\}$.

3. (Exercise 1.3.1 in [HK]) Let $T(x,y) = \left(\frac{2xy}{x+y}, \frac{x+y}{2}\right)$ for $x, y \in [0,\infty)$. As we discussed in class, T can be used to approximate square roots. To approximate $\sqrt{4}$, calculate $T^4(1,4)$ and determine how close the approximation obtained is to 2.

4. (Exercise 1.3.9 and 1.3.10 in [HK]) Consider the sequence $x_n = n^2$ for $n \in \mathbb{N} \cup \{0\}$. The sequence of last digits of $\{x_n\}_{n=0}^{\infty}$ turns out to be 0149656941 repeated periodically.

   (a) Prove that the sequence of last digits of $\{x_n\}_{n=0}^{\infty}$ indeed repeats periodically.

   (b) Explain why the part 01496569410 of the obtained sequence is a palindromic sequence, i.e., unchanged when reversed.

5. Let X be the set of bi-infinite sequences of 0's and 1's, i.e., if $x \in X$ then $x = \ldots x_{-n} \ldots x_{-1} x_0 x_1 \ldots x_n \ldots$ where $x_i \in \{0,1\}$ for all $i \in \mathbb{Z}$ (this set X is typically denoted $\{0,1\}^{\mathbb{Z}}$). For $x, y \in X$, we have $x = y$ if and only if $x_i = y_i$ for all $i \in \mathbb{Z}$.

   Define a function d on $X \times X$ by $d(x,y) = \frac{1}{n+1}$ where $n \in \mathbb{N} \cup \{0\}$ is such that $x_k = y_k$ if $|k| < n$ and ($x_n \neq y_n$ or $x_{-n} \neq y_{-n}$) when $x \neq y$. If $x = y$, then we set $d(x,y) = 0$.

   Show that d is a distance function, i.e., verify that $d(x,y) = 0$ if and only if $x = y$, $d(x,y) = d(y,x)$, and the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$.