Homework 1 - Solutions

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1. *Solution.* Let $S_n = \sum\limits_{i=0}^{n} b_i$. Then,

$$\begin{aligned} 2S_n &= 2\sum_{i=0}^{n} b_i = b_0 + \sum_{i=0}^{n-1}(b_i + b_{i+1}) + b_n \\ &= b_0 + \sum_{i=0}^{n-1} b_{i+2} + b_n = \sum_{i=0}^{n} b_i - b_1 + (b_{n+1} + b_n) \\ &= S_n - 1 + b_{n+2}. \end{aligned}$$

   Therefore, $S_n = b_{n+2} - 1$.

   □

2. *Solution.* The sequences of 0's and 1's of length 1 are 0 and 1 and they do not have two consecutive 0's, so $a_1 = 2$.

   The sequences of 0's and 1's of length 2 that do not have two consecutive 0's are 01, 10, and 11, so $a_2 = 3$.

   Let $n \in \mathbb{N}$. Any sequence of 0's and 1's of length $n+1$ that do not have two consecutive 0's that ends on 1 is a sequence of 0's and 1's of length n that do not have two consecutive 0's with added 1 on the $n+1$-st position. Any sequence of 0's and 1's of length $n+1$ that do not have two consecutive 0's that ends on 0 should have 1 on the n-th position. Thus, such sequence is obtained as a sequence of 0's and 1's of length n that do not have two consecutive 0's with added 1 on the n-th position and 0 on the $n+1$-st position. Therefore, $a_{n+1} = a_n + a_{n-1}$. □

3. *Solution.* We have

$$T(1,4) = (1.6, 2.5), \qquad T^2(1,4) = (1.95121951219, 2.05)$$
$$T^3(1,4) = (1.99939042974, 2.00060975609), \qquad T^4(1,4) = (1.99999990707, 2.00000009291).$$

   Notice that $2 - 1.99999990707 = 0.00000009293$ and $2.00000009291 - 2 = 0.00000009291$.

   Therefore, using $T^4(1,4)$ we can approximate $\sqrt{4}$ by 1.99999990707 which is at the distance 0.00000009293 to 2 or by 2.00000009291 which is at the distance 0.00000009291 to 2. □

4. Notice that for any $k \in \mathbb{N} \cup \{0\}$ we have that $k \equiv$ the last digit of $k \pmod{10}$.

   (a) *Solution.* For any $n \in \mathbb{N} \cup \{0\}$ we have $(n+10)^2 - n^2 = 10(2n+10)$ where $2n + 10 \in \mathbb{N}$ so $(n+10)^2 \equiv n^2 \pmod{10}$. Thus, the last digits of $(n+10)^2$ and n^2 are the same, so we see a periodic sequence. □

(b) *Solution.* Let $n \in \mathbb{N} \cup \{0\}$ such that $n \equiv 0 \pmod{10}$. Then, for any $k \in \{0, 1, 2, 3, 4, 5\}$, we have $(n+k)^2 - (n+10-k)^2 \equiv 4nk \pmod{10}$ so $(n+k)^2 - (n+10-k)^2 \equiv 0 \pmod{10}$ so $(n+k)^2 \equiv (n+10-k)^2 \pmod{10}$. Therefore, we see a palindromic sequence 01496569410. □

5. *Solution.* First, we show that $d(x, y) = 0$ if and only if $x = y$. If $x = y$, then by definition we have that $d(x, y) = 0$. Assume $x \neq y$. Then there exists $k \in \mathbb{Z}$ such that $x_k \neq y_k$. In particular, there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_k = y_k$ if $|k| < n$ and ($x_n \neq y_n$ or $x_{-n} \neq y_{-n}$). Thus, $d(x, y) = \frac{1}{n+1} \neq 0$. Therefore, $d(x, y) = 0$ implies $x = y$.

   Second, we show that $d(x, y) = d(y, x)$. If $d(x, y) = 0$, then by what we already showed we have $x = y$ so $y = x$ and $d(y, x) = 0$. Assume that $d(x, y)\frac{1}{n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then, $x_k = y_k$ if $|k| < n$ and ($x_n \neq y_n$ or $x_{-n} \neq y_{-n}$) so $y_k = x_k$ if $|k| < n$ and ($y_n \neq x_n$ or $y_{-n} \neq x_{-n}$) so $d(y, x) = \frac{1}{n+1}$ by definition of d.

   Finally, we show the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$. If any of $d(x, z)$, $d(x, y)$, or $d(y, z)$ is equal to 0, then the statement is obviously true from the fact that $d(x, y) = 0$ if and only if $x = y$. Assume that $d(x, z)$, $d(x, y)$, and $d(y, z)$ are all non-zero.

   Let $d(x, y) = \frac{1}{l+1}$ and $d(y, z) = \frac{1}{m+1}$ for some $l, m \in \mathbb{N} \cup \{0\}$. Without loss of generality we can assume that $l \leq m$. Then, $x_k = y_k$ if $|k| < l$ and ($x_l \neq y_l$ or $x_{-l} \neq y_{-l}$). Also, $y_k = z_k$ if $|k| < m$ and ($y_m \neq z_m$ or $y_{-m} \neq z_{-m}$). Since $l \leq m$, we have that $x_k = z_k$ if $|k| < l$. Thus, if $x_i \neq z_i$ then $|i| \geq l$ and $d(x, z) \leq \frac{1}{l+1} \leq d(x, y) + d(y, z)$.

   Remark: We actually showed a stronger inequality that $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. □