Homework 2 - Solutions

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1. *Solution.* Since f is a contraction, there exists $0 \leq \lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

   Let $x \in X$ and $\varepsilon > 0$. If $\lambda = 0$, then $d(f(x), f(y)) \leq 0$ for all $x, y \in X$ so, for example, we can take $\delta = 1$ and for all $y \in X$ with $d(x, y) < \delta$ we have $d(f(x), f(y)) < \varepsilon$.

   Assume $\lambda \neq 0$. Let $\delta = \frac{\varepsilon}{\lambda}$. For all $y \in X$ such that $d(x, y) < \delta$ we have

   $$d(f(x), f(y)) \leq \lambda\delta < \lambda\frac{\varepsilon}{\lambda} = \varepsilon.$$

   Therefore, f is continuous on X. □

2. *Solution.* Let d be a distance function on $\mathbb{R}$.

   Since f is a contraction on X, there exists $\lambda \in [0, 1)$ such that $d(f(a), f(b)) \leq \lambda d(a, b)$ for all $a, b \in X$. Let x and y be fixed points of f, then $d(f(x), f(y)) = d(x, y)$ and $d(f(x), f(y)) \leq \lambda d(x, y)$ so $d(x, y)(1-\lambda) \leq 0$. Since $\lambda < 1$ and d is a distance function,we have $d(x, y)(1-\lambda) \geq 0$. As a result, $d(x, y) = 0$ so $x = y$. □

3. *Solution.* (a) Since $|\cos(3x)| \leq 1$, $|\sin(x)| \leq 1$, for $x \in [-1, 1]$ we have

   $$|f(x)| = \left|\frac{1}{6}\cos(3x) + \frac{1}{4}\sin(x) + \frac{1}{5}x\right| \leq \frac{1}{6}|\cos(3x)| + \frac{1}{4}|\sin(x)| + \frac{1}{5}|x| \leq \frac{1}{6} + \frac{1}{4} + \frac{1}{5} = \frac{37}{60} < 1$$

   so $f(x) \in I$ for all $x \in I$.

   (b) We have that $f'(x) = -\frac{1}{2}\sin(3x) + \frac{1}{4}\cos(x) + \frac{1}{5}$. We have $|f'(x)| \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{19}{20} < 1$ for $x \in I$. Thus, f is a contraction on I.

   □

4. *Solution.* (a) Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. We have

   $$\begin{aligned} d(f(x_1, x_2), f(y_1, y_2)) &= \sqrt{\left(\frac{x_1 - 1}{2} - \frac{y_1 - 1}{2}\right)^2 + \left(\frac{x_2 + 4}{3} - \frac{y_2 + 4}{3}\right)^2} \\ &= \sqrt{\frac{(x_1 - y_1)^2}{4} + \frac{(x_2 - y_2)^2}{9}} \\ &\leq \sqrt{\frac{(x_1 - y_1)^2}{4} + \frac{(x_2 - y_2)^2}{4}} \\ &\leq \frac{1}{2}\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= \frac{1}{2}d((x_1, x_2), d(y_1, y_2)). \end{aligned}$$

Thus, f is $\frac{1}{2}$-contraction.

(b) To find a fixed point we need to solve equation $f(x_1, x_2) = (x_1, x_2)$. We have

$$\left(\frac{x_1 - 1}{2}, \frac{x_2 + 4}{3}\right) = (x_1, x_2)$$
$$\Leftrightarrow \frac{x_1 - 1}{2} = x_1 \qquad \text{and} \qquad \frac{x_2 + 4}{3} = x_2$$
$$\Leftrightarrow x_1 = -1 \qquad \text{and} \qquad x_2 = 2.$$

Therefore, $x_0 = (-1, 2)$ is the unique fixed point of f.

(c) Let $x = (1, -1)$. We have $f(x) = f(1, -1) = (0, 1)$. In the item (1) we got $\lambda = \frac{1}{2}$. We have $d(f(x), x) = \sqrt{5}$. Using the hint, we want to find $n \in \mathbb{N}$ such that $\frac{\lambda^n}{1-\lambda} d(f(x), x) < 0.01$, i.e., $\left(\frac{1}{2}\right)^{n-1} < \frac{0.01}{\sqrt{5}}$. We want n such that $2^{n-1} > 100\sqrt{5}$ so $n > 1 + \frac{\ln(100\sqrt{5})}{\ln 2}$. Using calculator, we can see that we can take $n = 8$ as $8 > 1 + \frac{\ln(100\sqrt{5})}{\ln 2}$.

□

5. *Solution.* Let λ be any number between 1 and $|f'(p)|$, e.g., $|f'(p)| > \lambda = \frac{1+|f'(p)|}{2} > 1$.

Since $f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = \lim_{x \to p} \frac{f(x) - p}{x - p}$, there exists $\varepsilon > 0$ such that for all x such that $0 < |x - p| < \varepsilon$, we have $\left|\frac{f(x)-p}{x-p}\right| \geq \lambda$, i.e., $|f(x) - p| \geq \lambda |x - p|$.

Let x be such that $|x - p| = a \in (0, \varepsilon)$. Since $\lambda > 1$, there exists N such that $\lambda^N a > \varepsilon$. Assume $0 < |f^n(x) - p| < \varepsilon$ for $n = 1, 2, \ldots, N - 1$. (Otherwise, we have already found already a moment when an iterate of x is outside of ε-neighborhood of p. Notice that if $|x - p| \in (0, \varepsilon)$, then $|f(x) - p| \neq 0$.) Then, we have

$$|f^N(x) - p| \geq \lambda |f^{N-1}(x) - p| \geq \ldots \geq \lambda^N |x - p| = \lambda^N a > \varepsilon.$$

Therefore, f is a repeller. □

6. *Solution.* (a) To find fixed points, we need to solve $f(x) = x$. We obtain

$$x + \frac{1}{10}\cos(3\pi x) = x \quad \Leftrightarrow$$
$$\cos(3\pi x) = 0 \quad \Leftrightarrow$$
$$3\pi x = \frac{\pi}{2} + \pi n \text{ where } n \in \mathbb{Z} \quad \Leftrightarrow$$
$$x_n = \frac{1}{6} + \frac{n}{3} \text{ where } n \in \mathbb{Z}.$$

Therefore, x_n for $n = 0, 1, 2$ are the fixed points in $[0, 1]$, i.e., $\frac{1}{6}$, $\frac{1}{2}$ and $\frac{5}{6}$ are the fixed points of f on $[0, 1]$.

(b) We have $f'(x_n) = 1 - \frac{3\pi}{10}\sin(3\pi x)$ so $f'(\frac{1}{6} + \frac{n}{3}) = 1 - \frac{3\pi}{10}\sin(\frac{\pi}{2} + \pi n)$. Thus, we obtain $f'(x_{2k}) = 1 - \frac{3\pi}{10} \in (0, 1)$ and $f'(x_{2k+1}) = 1 + \frac{3\pi}{10} > 1$. Therefore, $x_0 = \frac{1}{6}$, $x_2 = \frac{5}{6}$ are attracting fixed points and $x_1 = \frac{1}{2}$ are repelling fixed points on $[0, 1]$.

□