Homework 2

MAT 351, Instructor: Alena Erchenko

1. Let (X, d) be a metric space. Show that if $f\colon X \to X$ is a contraction map, then f is continuous.

   Recall that $f\colon X \to X$ is continuous at $x \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in X$ with $d(x, y) < \delta$ then $d(f(x), f(y)) < \varepsilon$. We say f is continuous on X if f is continuous at all $x \in X$.

2. Let X be a subset in $\mathbb{R}$. Consider a contraction $f\colon X \to X$. Prove that f cannot have more than one fixed point, meaning, if $f(x) = x$ and $f(y) = y$, then $x = y$.

3. Let $f(x) = \frac{1}{6}\cos(3x) + \frac{1}{4}\sin(x) + \frac{1}{5}x$ for all $x \in I$ where $I = [-1, 1]$.

   (a) Prove that f maps I to I.

   (b) Prove that f is a contraction on I.

4. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x_1, x_2) = \left(\frac{x_1-1}{2}, \frac{x_2+4}{3}\right)$ for $(x_1, x_2) \in \mathbb{R}^2$. Let $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ be the distance function on $\mathbb{R}^2$.

   (a) Find $\lambda < 1$ such that f is a λ-contraction and prove that the λ that you found works.

   (b) Find the fixed point x_0 of f.

   (c) Let $x = (1, -1)$. Find $n \in \mathbb{N}$ such that $d(f^n(x), x_0) < 0.01$. Show your work.
   Hint: From the proof of the contraction principle, if f is λ-contraction on $\mathbb{R}^2$ then you have that for any $x \in \mathbb{R}^2$ and $n \in \mathbb{N}$ we have $d(f^n(x), x_0) \leq \frac{\lambda^n}{1-\lambda} d(f(x), x)$.

5. Let I be an interval. Consider a differentiable function $f\colon I \to I$ with a fixed point p. Show that if $|f'(p)| > 1$ then p is a repeller.

6. Let $f(x) = x + \frac{1}{10}\cos(3\pi x)$ for $x \in [0, 1]$.

   (a) Find the fixed points of f.

   (b) Determine which fixed points are attracting and which are repelling.