Homework 3 - Solutions

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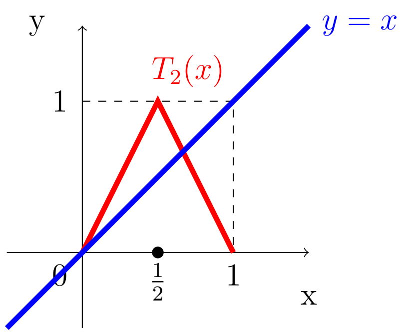
1. *Solution.*



Note that x is a periodic point of T_2 of period n iff $T_2^n(x) = x$. Moreover, $T_2([0, \frac{1}{2}]) = [0, 1]$, $T_2([\frac{1}{2}]) = [0, 1]$, $T_2(0) = T_2(1) = 0$, and $T_2(\frac{1}{2}) = 1$. Recall that fixed points of a function $y = f(x)$ are given by the intersection points with the line $y = x$. Each "tooth" like on the graph gives 2 fixed points. Let w_n be the number of "teeth" for T_2^n. Then, $w_1 = 1$ and $w_{n+1} = 2w_n$ from the properties of T_2 so $w_n = 2^{n-1}$. Therefore, T_2^n has 2^n fixed points so T_2 has 2^n periodic points of period n. □

2. *Proof.* ($\Rightarrow$): Since $T_2(x) = 2x$ or $2 - 2x$, we have that $T_2^n(x) = k + (-1)^l 2^n x$ for some $k \in \mathbb{Z}$ and $l \in \{0, 1\}$. Thus, $T_2^n(x) = T_2^m(x)$ implies that $k_1 + (-1)^{l_1} 2^n x = k_2 + (-1)^{l_2} 2^m x$ for some $k_1, k_2 \in \mathbb{Z}$ and $l_1, l_2 \in \{0, 1\}$. We obtain that if $m \neq n$ and $T_2^n(x) = T_2^m(x)$, then $x = \frac{k_1 - k_2}{(-1)^{l_2} 2^m - (-1)^{l_1} 2^n} \in \mathbb{Q}$. Thus, if x is eventually periodic for T_2 then $x \in [0, 1] \cap \mathbb{Q}$.

   ($\Leftarrow$): Let $x \in [0, 1] \cap \mathbb{Q}$. Then, $x = \frac{p}{q}$ where $q \in \mathbb{N}$ and $p \in \{0, 1, \ldots, q\}$. Also, $T(\frac{a}{q}) = \frac{2a}{q}$ or $T(\frac{a}{q}) = 2 - 2\frac{a}{q} = \frac{2(q-a)}{q}$. In any case, $T(\frac{a}{q})$ is of the form $\frac{b}{q}$ where $b \in \{0, 1, 2, \ldots, q\}$. Therefore, $T^n(x) = T^n(\frac{p}{q}) = \frac{k}{q}$ for some $k = \{0, 1, \ldots, q\}$. Since there are only finitely many options for k, we obtain that there exists $n, m \in \mathbb{N} \cup \{0\}$ such that $n \neq m$ and $T^n(x) = T^m(x)$ for $x \in \mathbb{Q}$ so x is eventually periodic.

   □

3. *Solution.* Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume, for contradiction, that R_α has a periodic point $x \in S^1$ with period $n \in \mathbb{N}$. Then, $R_\alpha^n(x) = x$ so $x + n\alpha = x + k$ for some $k \in \mathbb{Z}$. Therefore, $\alpha = \frac{k}{n}$ where $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ so $\alpha \in \mathbb{Q}$ contradicting the fact that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. □

4. (a) *Solution.* Let $x \in S^1$. $E_2(x) = x \Leftrightarrow 2x = x \pmod 1 \Leftrightarrow 2x = x + k$ for some $k \in \mathbb{Z} \Leftrightarrow x = k$ for some $k \in \mathbb{Z}$. Thus, $0 \in S^1$ is a unique fixed point. □

   (b) *Solution.* Let $x \in S^1$. $E_2^n(x) = x \Leftrightarrow 2^n x = x \pmod 1 \Leftrightarrow 2^n x = x + k$ for some $k \in \mathbb{Z} \Leftrightarrow x = \frac{k}{2^n - 1}$ for some $k \in \mathbb{Z}$.

   Thus, $x_k = \frac{k}{2^n - 1} \in S^1$ where $k = 0, 1, 2, \ldots, 2^n - 2$ are all periodic points of period n for E_2. □

(c) *Solution.* By the part (b), all periodic points of period 3 for E_2 on S^1 are $x_k = \frac{k}{7}$ for $k = 0, 1, \ldots, 6$.

The points has prime period 3 if it is not a periodic point with period 2 or a fixed point. Periodic points with period 2 are $y_m = \frac{m}{3}$ for $m = 0, 1, 2$ and the only fixed point is 0. Therefore, the periodic points of E_2 on S^1 with prime period 3 are $x_k = \frac{k}{7}$ for $k = 1, \ldots, 6$. □

(d) *Proof.* By the part (b), we have that $x_{k,n} = \frac{k}{2^n-1} \in S^1$ where $k = 0, 1, 2, \ldots, 2^n - 2$ are all periodic points of period n for E_2. Notice that for fixed n, $x_{k,n}$ split S^1 into $2^n - 1$ half-open arcs of length $\frac{1}{2^n-1}$. We have that $\lim_{n\to\infty} \frac{1}{2^n-1} = 0$ so for any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that $\frac{1}{2^N-1} < \varepsilon$. Thus, since any $z \in S^1$ belongs to the one of arcs of length $\frac{1}{2^N-1} < \varepsilon$ with endpoints being periodic points of E_2, we obtain that there exists a periodic point $y \in S^1$ of E_2 such that $d(z, y) < \varepsilon$. □

5. (a) *Solution.*

$$\sum_{d=1}^{9} f(d) = \sum_{d=1}^{9} \log_{10}\left(\frac{d+1}{d}\right) = \log_{10}\left(\prod_{d=1}^{9}\left(\frac{d+1}{d}\right)\right) = \log_{10}(10) = 1.$$

Other way,

$$\sum_{d=1}^{9} f(d) = \sum_{d=1}^{9} \lim_{n\to\infty} \frac{F_d(n)}{n} = \lim_{n\to\infty} \frac{\sum_{d=1}^{9} F_d(n)}{n} = \lim_{n\to\infty} \frac{n}{n} = \lim_{n\to\infty} 1 = 1.$$

□

(b) *Solution.*

$$f(1) = \log_{10}\left(\frac{2}{1}\right) \approx 0.301, \qquad f(2) = \log_{10}\left(\frac{3}{2}\right) \approx 0.176, \qquad f(3) = \log_{10}\left(\frac{4}{3}\right) \approx 0.125,$$
$$f(4) = \log_{10}\left(\frac{5}{4}\right) \approx 0.097, \qquad f(5) = \log_{10}\left(\frac{6}{5}\right) \approx 0.079, \qquad f(6) = \log_{10}\left(\frac{7}{6}\right) \approx 0.067,$$
$$f(7) = \log_{10}\left(\frac{8}{7}\right) \approx 0.058, \qquad f(8) = \log_{10}\left(\frac{9}{8}\right) \approx 0.051, \qquad f(9) = \log_{10}\left(\frac{10}{9}\right) \approx 0.046.$$

□

(c) *Solution.* Let $g(2)$ be the asymptotic frequency of 2 being the second digit of 2^n. Since we look at second digit, we can have any natural number from 1 to 9 as a first digit. Thus,

$$\begin{aligned} g(2) &= f(12) + f(22) + f(32) + f(42) + f(52) + f(62) + f(72) + f(82) + f(92) \\ &= \log_{10}\left(\frac{13}{12}\right) + \log_{10}\left(\frac{23}{22}\right) + \log_{10}\left(\frac{33}{32}\right) + \log_{10}\left(\frac{43}{42}\right) + \log_{10}\left(\frac{53}{52}\right) \\ &+ \log_{10}\left(\frac{63}{62}\right) + \log_{10}\left(\frac{73}{72}\right) + \log_{10}\left(\frac{83}{82}\right) + \log_{10}\left(\frac{93}{92}\right) \\ &\approx 0.109 \end{aligned}$$

□

(d) *Solution.* Let $d \in \{1,2,3,4,5,6,7,8,9\}$. Then, d is the first digit of $3 \cdot 2^n$ iff $d \cdot 10^k \leq 3 \cdot 2^n \leq (d+1) \cdot 10^k$ for some $k \in \mathbb{N} \cup \{0\}$ iff $\log_{10} d + k \leq \log_{10} 3 + n \log_{10} 2 < \log_{10}(d+1) + k$ for some $k \in \mathbb{N} \cup \{0\}$.

Thus, d gives the first digit of $2^n \Leftrightarrow$ if we start with $\log_{10} 3$ and apply the rotation R_α with $\alpha = \log 2$ n times, the point will end up in the interval $[\log_{10} d, \log_{10}(d+1))$ on S^1.

Since $\alpha = \log 2$ is irrational, then the positive semiorbit of any $x \in S^1$ under R_α is uniformly distributed. Therefore, the asymptotic frequency of d as the first digit for the numbers of the form $3 \cdot 2^n$ is the same as the asymptotic frequency of d as the first digit for the numbers of the form 2^n. □