Homework 3

MAT 351, Instructor: Alena Erchenko

[D] stands for "An introduction to chaotic dynamical systems" by R.L. Devaney

1. (Exercise 3 in Chapter 1.5 in [D]) Sketch the graph of the tent map
$$T_2(x) = \begin{cases} 2x & \text{if} \quad 0 \le x \le \dfrac{1}{2}, \\ 2-2x & \text{if} \quad \dfrac{1}{2} \le x \le 1 \end{cases}$$
on the unit interval $[0,1]$. Use the graph of T_2^n to conclude that T_2 has exactly 2^n periodic points of period n.

2. Let $T_2\colon [0,1] \to [0,1]$ be the tent map from the previous exercise. We say that x is eventually periodic for T_2 if $T_2^n(x) = T_2^m(x)$ for some $m, n \in \mathbb{N} \cup \{0\}$ such that $m \neq n$.

   Show that x is eventually periodic for T_2 if and only if $x \in [0,1] \cap \mathbb{Q}$.

3. Suppose that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Prove that the rotation map R_α on $S^1 = \mathbb{R}/\mathbb{Z}$ has no periodic points.

4. On $S^1 = \mathbb{R}/\mathbb{Z}$ define a map E_2 by $E_2(x) = 2x \pmod 1$ for all $x \in S^1$.
   (a) Find all fixed points for E_2.
   (b) Given $n \in \mathbb{N}$, find all periodic points of period n.
   (c) Find all periodic points of prime period 3.
   (d) Prove that periodic points of E_2 are dense in S^1.

5. Consider $d \in \mathbb{N}$. Let $F_d(n)$ be the number of integers $k \in [0, n)$ such that d gives the first digits of 2^k. Then, the asymptotic frequency $f(d)$ of d defined by $f(d) = \lim\limits_{n\to\infty} \frac{F_d(n)}{n}$ is equal to $\log_{10}\left(\frac{d+1}{d}\right)$ by a theorem proved in the class.
   (a) Verify that $\sum\limits_{d=1}^{9} f(d) = 1$.
   (b) Use a calculator to compute the asymptotic frequencies $f(d)$ for $d = 1, 2, \ldots, 9$ (write your answer up to three decimal places).
   (c) Find the asymptotic frequency of 2 being the second digit of 2^n. Explain your answer. Use a calculator to obtain a numerical answer (write your answer up to three decimal places).
   Hint: Can you say something about the first two digits?
   (d) Find the asymptotic frequencies of $1, 2, 3, \ldots, 9$ as the first digits for the numbers of the form $3 \cdot 2^n$ where $n = 0, 1, 2, \ldots$. Are they different from those for 2^n? Justify your answer.