Homework 4 - Solutions

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1. (a) *Solution.* Notice that $\frac{19}{27} = \frac{2\cdot 3^2+1}{3^3} = \frac{2}{3} + \frac{1}{3^3}$. Thus, $\frac{19}{27} = 0.202222\ldots = 0.210000\ldots$. ☐

   (b) *Solution.* $\frac{1}{3} \in S^1$ is neither periodic nor dense as $E_3^n(1/3) = 0 \neq \frac{1}{3}$ for all $n \in \mathbb{N}$. ☐

   (c) *Proof.* ($\Rightarrow$) Assume $x \in S^1$ is eventually periodic for E_3. Then, there exists $n > m \geq 0$ such that $E_3^n(x) = E_3^m(x)$ so $3^n x - 3^m x = k$ for some $k \in \mathbb{Z}$ so $x = \frac{k}{3^n - 3^m} \in \mathbb{Q}$.
   ($\Leftarrow$) Let $x \in \mathbb{Q} \cap [0,1)$. Then, $x = \frac{p}{q}$ where $q \in \mathbb{N}$ and $p \in \{0,1,2,\ldots,q-1\}$. We have $E_3(\frac{a}{q}) = \frac{3a}{q} \pmod 1$ so it has the same form $\frac{b}{q}$ where $b \in \{0,1,2,\ldots,q-1\}$. Since there are only finitely many such fractions, there exist $n, m \in \mathbb{Z}$ such that $n > m \geq 0$ and $E_3^n(x) = E_3^m(x)$. ☐

   (d) *Solution.* Let $x = 0$ and $y = \frac{1}{3}$, then $d(x,y) = \frac{1}{3}$, $E_3(x) = 0$, $E_3(y) = 0$, and $d(E_3(x), E_3(y)) = 0$. ☐

   (e) *Proof.* Let $x = 0.a_1a_2a_3\ldots a_n x_{n+1}x_{n+2}\ldots$ and $y = 0.a_1a_2a_3\ldots a_n y_{n+1}y_{n+2}\ldots$ in base 3. Then,
$$|x-y| = \sum_{k=n+1}^{\infty} \frac{|x_k - y_k|}{3^k} \leq \sum_{k=n+1}^{\infty} \frac{2}{3^k} = 2\frac{3^{-(n+1)}}{1-\frac{1}{3}} = \frac{1}{3^n}.$$
   Thus, by the definition of d, we have $d(x,y) \leq \frac{1}{3^n}$. ☐

   (f) *Proof.* Consider an interval $I \subset S^1$. Since the length of I increases by a factor of 3 each time we apply E_3 until the length of the interval to which we apply E_3 is less than $\frac{1}{3}$ and the image under E_3 coincides with S^1 if the length of an interval is greater or equal to $\frac{1}{3}$, there exists $n \in \mathbb{N}$ such that $E_3^n(I) = S^1$. If $E_3(I) = I$, then $E_3^n(I) = I = S^1$. Thus, there is no an interval $I \subset S^1$ such that $E_3(I) = I$ and $I \neq S^1$. ☐

2. (a) *Proof.* Since $f\colon [0,1] \to [0,1]$ is surjective, then there exists $a, b \in [0,1]$ such that $f(a) = 0$ and $f(b) = 1$, in particular, $f(a) \leq a$ and $f(b) \geq b$. Consider a function $g(x) = f(x) - x$ on $[0,1]$. The function g is continuous on $[0,1]$ because f is continuous on $[0,1]$. Moreover, $g(a) \leq 0$ and $g(b) \geq 0$. Thus, by the Intermediate Value Theorem, there exists $c \in [a,b] \subset [0,1]$ such that $g(c) = 0$ so $f(c) = c$ so c is a fixed point for f. ☐

   (b) *Proof.* First, we show that either f or f^2 must have points $0 \leq x_1 < x_2 \leq 1$ such that x_1 maps to 0 and x_2 maps to 1. Since $f\colon [0,1] \to [0,1]$ is surjective, then there exists $a, b \in [0,1]$ such that $f(a) = 0$ and $f(b) = 1$. Notice that $a \neq b$ since f is a function. If $a < b$, then let $x_1 = a$ and $x_2 = b$. Assume that there are no $0 \leq x_1 < x_2 \leq 1$ such that $f(x_1) = 0$ and $f(x_2) = 1$. Let $m_1 = \max\{x | f(x) = 1\}$ and $m_2 = \min\{x | f(x) = 0\}$. By our assumption $m_1 < m_2$. We claim that there exists $x_1, x_2 \in [m_1, m_2]$ such that $x_1 < x_2$ and $f^2(x_1) = 0$ and $f^2(x_2) = 1$. By the Intermediate Value Theorem applied to f, we have that there exists $x_1 \in [m_1, m_2]$ such that $f(x_1) = m_2$ because $f(m_2) = 0$ and $f(m_1) = 1$.

Thus, $f^2(x_1) = f(m_2) = 0$. Since $0 \geq m_1 < m_2$, $f(x_1) = m_2$ and $f(m_2) = 0$, we have that there exist $x_2 \in [x_1, m_2]$ such that $f(x_2) = m_1$ so $f^2(x_2) = f(m_1) = 1$. In particular, $0 \leq x_1 < x_2 \leq 1$ by construction and the fact that f^2 is a function.

Now we prove a fact that if $g\colon [0,1] \to [0,1]$ is a continuous surjective function such that there exist x_1, x_2 such that $0 \leq x_1 < x_2 \leq 1$ and $g(x_1) = 0$ and $g(x_2) = 1$ then g has at least two fixed points.

Case 1: Assume $x_1 = 0$ and $x_2 = 1$. Then, $g(0) = 0$ and $g(1) = 1$ so g has at least two fixed points.

Case 2: Assume $x_1 = 0$ and $x_2 < 1$. Then, $g(0) = 0$ and $g(1) < 1$. Moreover, since $g(x_2) = 1 > x_2$ and $g(1) < 1$, by considering function $g(x) - x$ and applying the Intermediate Value theorem, there exists $c \in (x_2, 1) \subset (0,1)$ such that $g(c) = c$. Thus, g has at least two fixed points.

Case 3: Assume $x_1 > 0$. Without loss of generality we can assume $g(0) \neq 0$ as otherwise we could pick $x_1 = 0$ and just follow Cases 1 and 2. If $g(0) \neq 0$, then $g(0) > 0$ and $g(x_1) = 0 < x_1$, so, by considering function $g(x) - x$ and applying the Intermediate Value theorem, there exists $c \in (0, x_1) \subset (0,1)$ such that $g(c) = c$. Moreover, since $g(x_1) = 0 < x_1$ and $g(x_2) = 1 \geq x_2$ and $x_1 < x_2$, by considering function $g(x) - x$ and applying the Intermediate Value theorem, there exists $d \in (x_1, x_2] \subset (0,1]$ such that $g(d) = d$. By construction, $c \neq d$ so g has at least two fixed points.

As a result, we have that f^2 has at least two fixed points as, by the first step:

i. either f has the properties of the described above function g, and f has at least two fixed points which implies that f^2 has at least two fixed points;

ii. or f^2 can has the properties of the described above function g so f^2 has at least two fixed points.

□

3. (a) *Solution.* The total length of removed intervals is

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

□

(b) *Proof.* We say $x = 0.x_1x_2\ldots$ in the base 4 if $x = \sum\limits_{n=1}^{\infty} \frac{x_n}{4^n}$ where $x_n \in \{0,1,2,3\}$ for all n.

Then, $x \in K(4)$ if and only if $x = 0.x_1x_2\ldots$ in the base 4 where $x_n \in \{0,3\}$ for all n. □

(c) *Solution.* $\frac{1}{5} = \sum\limits_{n=1}^{\infty} \frac{3}{4^{2n}} = 0.03030303.......$ in base 4. Thus, $\frac{1}{5} \in K(4)$.

$\frac{17}{21} = \sum\limits_{n=0}^{\infty} \left(\frac{3}{4^{3n+1}} + \frac{3}{4^{3n+3}}\right) = 0.303303303303.......$ in base 4. Thus, $\frac{17}{21} \in K(4)$.

□

4. *Solution.* (a) No. Let $m \in \mathbb{N} \setminus \{1\}$. For example, let $x = (1,1,1,\ldots)$ and $y = (1,m,m,\ldots)$ then $d(x,y) = |1-1| = 0$ but $x \neq y$.

(b) No. Let $m \in \mathbb{N} \setminus \{1\}$. For example, let $x = (1, 1, 1, \ldots)$ and $y = (m, m, m, \ldots)$ then $d(x, y) = \sum_{k=0}^{\infty} |1 - m| \notin [0, \infty)$ as the sum diverges because each term is $(m - 1) \geq 1$.

(c) Yes.

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