Homework 4

MAT 351, Instructor: Alena Erchenko

[ASY] stands for "Chaos: An introduction to dynamical systems" by Alligood, Sauer, and Yorke.

1. Let E_3 be the times-3 map on the circle $S^1 = \mathbb{R}/\mathbb{Z}$.
   (a) Give two expansions of the number $\frac{19}{27}$ in base 3.
   (b) Find a point whose orbit under the map E_3 is neither periodic nor dense. Explain your answer.
   (c) Show that a point x is eventually periodic (see previous homework for the definition) for the map E_3 if and only if $x \in \mathbb{Q} \cap S^1$.
   (d) Find a pair of points on S^1 whose distance is not tripled by E_3. Explain your answer.
   (e) Assume that $x, y \in [0, 1)$ have expansions in base 3 that coincide up to n-th place, then $d(x, y) \leq \frac{1}{3^n}$ where d is the distance on S^1.
   (f) Show that there is no an interval $I \subset S^1$ such that $E_3(I) = I$ and $I \neq S^1$.

2. (Exercise 3.11 in [ASY]) Let $f\colon [0, 1] \to [0, 1]$ be a surjective continuous map.
   (a) Prove that f must have at least one fixed point.
   Hint: Recall the Intermediate Value Theorem that can be used without proof:
   Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then, for any u such that $\min\{f(a), f(b)\} \leq u \leq \max\{f(a), f(b)\}$ *there exists $c \in [a, b]$ such that $f(c) = u$.*
   (b) Prove that f^2 must have at least two fixed points.
   Hint: Explain why either f or f^2 must have points $0 \leq x_1 < x_2 \leq 1$ such that x_1 maps to 0 and x_2 maps to 1.

3. (Exercise 4.2 in [ASY]) Consider the **middle-half Cantor set** $K(4)$ formed by deleting the middle half of each subinterval instead of the middle third (for example: On the first step you delete $(\frac{1}{4}, \frac{3}{4})$, on the second step you delete $(\frac{1}{16}, \frac{3}{16})$ and $(\frac{13}{16}, \frac{15}{16})$ and so on).
   (a) What is the total length of the subintervals removed from $[0, 1]$?
   (b) What numbers in $[0, 1]$ belong to $K(4)$?
   (c) Show that $\frac{1}{5}$ is in $K(4)$. What about $\frac{17}{21}$?

4. For each of the functions below, determine weather it is a distance function on Ω_m^+. If it is a distance, answer Yes. If it is not, answer *No* and explain why not.
   (a) $d(x, y) = |x_0 - y_0|$ where $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$;

(b) $d(x,y) = \sum_{k=0}^{\infty} |x_k - y_k|$ where $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$;

(c) $d(x,y) = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{3^k}$ where $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$.