Homework 5

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In the problems below consider Ω_m^+ (the space of all one-sided infinite sequences of symbols from $\{1, 2, \ldots, m\}$) with the distance function d given by $d(x, y) = 2^{-n}$ where $n \in \mathbb{N} \cup \{0\}$ is such that $x_k = y_k$ if $k < n$ and $x_n \neq y_n$ and $d(x, y) = 0$ if $x = y$. Here $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$. Denote by σ^+ the shift on Ω_m^+.

1. Give an example of a periodic point $\omega \in \Omega_3^+$ such that $d(\omega, \omega') < \frac{1}{32}$ for
$$\omega' = (3, 1, 3, 3, 1, 3, 3, 3, 1, 3, 3, 3, 3, 1, \ldots).$$
Explain you answer.

2. (a) Find a constant $C > 1$ such that $d(\sigma^+(\omega), \sigma^+(\omega')) \geq C d(\omega, \omega')$ for all $\omega, \omega' \in \Omega_m^+$ with $d(\omega, \omega') \leq \frac{1}{2}$. Justify your answer.
Remark: This is why we can say that σ^+ is expanding.
   (b) Is it true that $d(\sigma^+(\omega), \sigma^+(\omega')) > d(\omega, \omega')$ for all $\omega, \omega' \in \Omega_m^+$? Justify your answer.

3. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and consider the corresponding subshift of finite type (Ω_A^+, σ^+).
   (a) Draw the directed graph Γ_A corresponding to A.
   (b) Find all fixed points.
   (c) By examining the graph, find all periodic points of prime period 2.
   (d) Find the number of admissible paths of length 3 from 3 to 1 in Γ_A.
   (e) Find the number of periodic points of period 3 using A^3.

4. Consider the quadratic family $f_\lambda(x) = \lambda x(1 - x)$. Recall that if $\lambda > 2 + \sqrt{5}$ then $[0, 1] \setminus \{x \in [0, 1] | f_\lambda(x) > 1\} = I_0 \cup I_1$ where I_0, I_1 are two closed intervals.
Prove that if $\lambda > 2 + \sqrt{5}$, then for all $x \in I_0 \cup I_1$ there exists $\mu > 1$ such that $|f'_\lambda(x)| \geq \mu$.

5. Let $f\colon A \to A$ and $g\colon B \to B$ be topologically conjugate maps. Show that there exists a bijection between the periodic points of period n of f and the periodic points of period n of g for all $n \in \mathbb{N}$.

6. Consider the quadratic family $f_\lambda(x) = \lambda x(1 - x)$ for $\lambda > 2 + \sqrt{5}$. Let $\Lambda = [0, 1] \setminus \left(\bigcup_{n=0}^{\infty} A_n \right)$ where
$$A_n = \{x \in [0, 1] \mid f_\lambda^i(x) \in [0, 1] \text{ for } 1 \leq i \leq n \text{ and } f_\lambda^{n+1}(x) \notin [0, 1]\}.$$
Show that f_λ has a dense orbit in Λ.

7. Let $Q_c(x) = x^2 + c$ for $x \in \mathbb{R}$. Prove that if $c < \frac{1}{4}$, there is a unique $\lambda > 1$ such that Q_c is topologically conjugate to $f_\lambda(x) = \lambda x(1 - x)$ via a map of the form $h(x) = ax + b$ where $a, b \in \mathbb{R}$, i.e., there exists a homeomorphism of the given form such that $h \circ Q_c = f_\lambda \circ h$.