Homework 5 - Solutions

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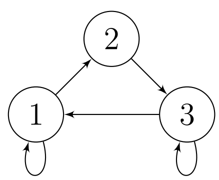
1. *Solution.* Notice that $32 = 2^5$. Thus, to have $d(\omega, \omega') < 2^{-5}$ we need that ω and ω' have the same entries up to the 5-th position including. In particular, for such ω we will have $d(\omega, \omega') \le 2^{-6}$. We choose a periodic point with that extra requirement. For example, take

$$\omega = (3, 1, 3, 3, 1, 3, 3, 1, 3, 3, 1, 3, 3, 1, 3, 3, 1, 3, \ldots).$$

$\square$

2. (a) *Solution.* Consider $\omega, \omega' \in \Omega_m^+$ such that $d(\omega, \omega') \le \frac{1}{2}$. If $\omega = \omega'$, then $d(\omega, \omega') = 0$ and $\sigma^+(\omega) = \sigma^+(\omega')$ so $d(\sigma^+(\omega), \sigma^+(\omega')) = 0$. Assume $\omega \ne \omega'$. Since $d(\omega, \omega') \le \frac{1}{2}$, there exists $n \in \mathbb{N}$ such that $\omega_k = \omega'_k$ if $k < n$ and $\omega_n \ne \omega'_n$. Then, $(\sigma^+(\omega))_k = (\sigma^+(\omega'))_k$ if $k < n-1$ and $(\sigma^+(\omega))_{n-1} \ne (\sigma^+(\omega'))_{n-1}$ so $d(\sigma^+(\omega), \sigma^+(\omega')) = 2^{-(n-1)} = 2 \cdot 2^{-n} = 2d(\omega, \omega')$. Therefore, we can take $C = 2$. $\square$

   (b) *Solution.* No. Let $m \in \mathbb{N} \setminus \{1\}$. Consider $\omega = (1, 1, 1, \ldots)$ and $\omega' = (m, 1, 1, \ldots)$. Then, $d(\omega, \omega') = 2^{-0} = 1$ and $\sigma^+(\omega) = \sigma^+(\omega')$ so $d(\sigma^+(\omega), \sigma^+(\omega')) = 0$. In particular, for our example, $d(\sigma^+(\omega), \sigma^+(\omega')) < d(\omega, \omega')$. $\square$

3. (a) *Solution.* 

$\square$

   (b) *Solution.* $(1, 1, 1, 1, \ldots)$ and $(3, 3, 3, 3, \ldots)$ are the only fixed points. $\square$

   (c) *Solution.* There are no periodic points with prime period 2. $\square$

   (d) *Solution.* We have $A^3 = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix}$. Since the $(3, 1)$-entry of the matrix A^3 is 3, there are 3 admissible paths of length 3 from 3 to 1. $\square$

   (e) *Solution.* The number of periodic points of period 3 is equal to $Trace(A^3) = 2 + 1 + 2 = 5$ (see A^3 in the previous item). $\square$

4. *Proof.* We have that $f_\lambda(x) > 1$ iff $\lambda x^2 - \lambda x + 1 < 0$ iff $\frac{1-\sqrt{1-\frac{4}{\lambda}}}{2} < x < \frac{1+\sqrt{1-\frac{4}{\lambda}}}{2}$. Therefore, $x \in I_0 \cup I_1$ means that $x \in [0, \frac{1-\sqrt{1-\frac{4}{\lambda}}}{2}] \cup [\frac{1+\sqrt{1-\frac{4}{\lambda}}}{2}, 1]$. In particular, $|1 - 2x| \ge \sqrt{1 - \frac{4}{\lambda}}$.

   Also, $f'_\lambda(x) = \lambda(1 - 2x)$ so if $x \in I_0 \cup I_1$ then $|f'_\lambda(x)| \ge \lambda\sqrt{1 - \frac{4}{\lambda}} > 1$ as $\lambda^2 - 4\lambda - 1 = (\lambda - (2 - \sqrt{5}))(\lambda - (2 + \sqrt{5})) > 0$ for $\lambda > 2 + \sqrt{5}$. Thus, we can take $\mu = \lambda\sqrt{1 - \frac{4}{\lambda}}$. $\square$

5. *Proof.* Since $f\colon A \to A$ and $g\colon B \to B$ are topologically conjugate, there exists a homeomorphism $h\colon A \to B$ such that $h \circ f = g \circ h$.

   Denote by $Per_n(f)$ and $Per_n(g)$ the set set of periodic points of period n for f and g, respectively. We define a map $\bar{h}\colon Per_n(f) \to Per_n(g)$ by $\bar{h}(p) = h(p)$ for $p \in Per_n(f)$. Let us show that $\bar{h}$ is well defined. Let $p \in Per_n(f)$, then $f^n(p) = p$. Thus, using $h \circ f = g \circ h$, we obtain that $h(p) = h(f^n(p)) = g \circ h(f^{n-1}(p)) = g^2 \circ h(f^{n-2}(p)) = \ldots = g^n \circ h(p) = g^n(h(p))$. Therefore, $g^n(h(p)) = h(p)$, i.e., $h(p) \in Per_n(g)$, so the codomain of $\bar{h}$ is $Per_n(g)$. Since h is invertible, $\bar{h}$ is invertible with $\bar{h}^{-1}(q) = h^{-1}(q)$ for $q \in Per_n(g)$ so $\bar{h}$ is a bijection. □

6. *Proof.* We showed in class that (Λ, f_λ) for $\lambda > 2 + \sqrt{5}$ is topologically conjugate to (Ω_2^+, σ^+) so there exists a homeomorphism $h\colon \Omega_2^+ \to \Lambda$ such that $f_\lambda \circ h = h \circ \sigma^+$.

   Let $\omega \in \Omega_2^+$. Then, $h((\sigma^+)^n)(\omega) = (f_\lambda)^n(h(\omega))$ for all $n \in \mathbb{N} \cup \{0\}$ so the orbit of ω is mapped to the orbit of $h(\omega) \in \Lambda$. Recall that the orbit of $\omega = (0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, \ldots, 1, 1, 1, \ldots)$ under σ^+ is dense in Ω_2^+. We show that the orbit of $h(\omega)$ under f_λ is dense in Λ.

   Let $z \in \Lambda$ and $\varepsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that $|f_\lambda^N(h(\omega)) - z| < \varepsilon$. Since h is homeomorphism, h^{-1} exists and h is continuous. In particular, there exists $\delta > 0$ such that if $x \in \Omega_2^+$ and $d(x, h^{-1}(z)) < \delta$ then $|h(x) - h(h^{-1}(z))| < \varepsilon$, i.e., $|h(x) - z| < \varepsilon$. From the density of the orbit of ω in Ω_2^+, there exists $N \in \mathbb{N}$ such that $d((\sigma^+)^N(\omega), h^{-1}(z)) < \delta$. Thus, $|h((\sigma^+)^N(\omega)) - z| < \varepsilon$ so $|f_\lambda^N(h(\omega)) - z| < \varepsilon$.

   As a result, the orbit of $h(\omega)$ is dense in Λ.

   □

7. *Proof.* Since h has to be homeomorphism, in particular, invertible, we should have $a \neq 0$. Let us show that if $x < \frac{1}{4}$, we can find $\lambda > 1$, $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that $h \circ Q_c(x) = f_\lambda \circ h(x)$ for all $x \in \mathbb{R}$.

   We have $h \circ Q_c(x) = f_\lambda \circ h(x)$ for all $x \in \mathbb{R}$ iff $a(x^2 + c) + b = \lambda(ax + b)(1 - (ax + b))$ for all $x \in \mathbb{R}$ iff $a = -\lambda a^2$ and $-2\lambda ab + \lambda a = 0$ and $ac + b = -\lambda b^2 + \lambda b$. Since $a \neq 0$, we have $a = -\frac{1}{\lambda}$ (in particular, $\lambda \neq 0$) and $b = \frac{1}{2}$. Using $ac + b = -\lambda b^2 + \lambda b$, we obtain that we need $\lambda^2 - 2\lambda + 4c = 0$. Since $c < \frac{1}{4}$, the solutions of the previous equation are $\lambda_1 = 1 + \sqrt{1 - 4c}$ and $\lambda_2 = 1 - \sqrt{1 - 4c}$ such that $\lambda_1 > 1$ and $\lambda_2 < 1$. Therefore, there exists a unique $\lambda = 1 + \sqrt{1 - 4c} > 1$ such that Q_c is topologically conjugate to f_λ via a homeomorphism h defined by $h(x) = -\frac{1}{\lambda}x + \frac{1}{2}$. □