Homework 10

MAT 351, Instructor: Alena Erchenko

1. Let $X \subset \mathbb{R}^k$ and μ be the Lebesgue measure on X.

   (a) Let $s\colon X \to \mathbb{R}$ be a simple function defined by $s(x) = \sum\limits_{k=1}^{n} a_k \chi_{B_k}(x)$ where $a_k \in \mathbb{R}$ and B_k is measurable for all $k \in \{1, 2, \ldots, n\}$ and $B_i \cap B_j = \emptyset$ if $i \neq j$. Show that if $s(x) \geq 0$ for all $x \in X$, then $\int_X s(x)d\mu(x) \geq 0$.
   <u>Remark:</u> In particular, this implies that if $f\colon X \to \mathbb{R}$ is a measurable and integrable function such that $f(x) \geq 0$ for all $x \in X$, then $\int_X f(x)d\mu(x) \geq 0$.

   (b) Let $s\colon X \to \mathbb{R}$ and $h\colon X \to \mathbb{R}$ be simple functions. Show that for all $a, b \in \mathbb{R}$, we have

   $$\int_X (as(x) + bh(x))d\mu(x) = a\int_X s(x)d\mu(x) + b\int_X h(x)d\mu(x).$$

   <u>Remark:</u> In particular, it is possible to obtain, that for any measurable integrable functions $f\colon X \to \mathbb{R}$ and $g\colon X \to R$ and any $a, b \in \mathbb{R}$, we have

   $$\int_X (af(x) + bg(x))d\mu(x) = a\int_X f(x)d\mu(x) + b\int_X g(x)d\mu(x).$$

   (c) Let $f\colon X \to \mathbb{R}$ be a measurable integrable function. Show that

   $$\left|\int_X f(x)d\mu(x)\right| \leq \int_X |f(x)|d\mu(x).$$

   You can use the remarks from the previous items.

2. Let $X \subset \mathbb{R}^k$ and μ be a measure on X such that $\mu(X) = 1$. Consider a measure-preserving map $T\colon X \to X$. Assume that for any measurable function $f\colon X \to \mathbb{R}$ such that $f \circ T = f$ almost every where, we have f is constant almost everywhere. Show that for any measurable set $A \subset X$ such that $T^{-1}(A) = A$, we have $\mu(A) = 0$ or $\mu(A) = 1$.

   <u>Remark:</u> We show in this problem that one definition of ergodic map implies the other definition.

3. Let $X \subset \mathbb{R}^k$ such that $\mu(X) = 1$, where μ is the Lebesgue measure on X. Let $A_1, A_2, A_3 \subset X$ be measurable sets. Show that

   $$\mu(A_1 \Delta A_3) \leq \mu(A_1 \Delta A_2) + \mu(A_2 \Delta A_3),$$

   where $A_i \Delta A_j = (A_i \setminus A_j) \cup (A_j \setminus A_i)$.
   <u>Hint:</u> Show that $\mu(A_i \Delta A_j) = \int_X |\chi_{A_i}(x) - \chi_{A_j}(x)|d\mu(x)$.

4. Let $X \subset \mathbb{R}^k$ and μ be a measure on X such that $\mu(X) = 1$. Consider an ergodic measure-preserving map $T\colon X \to X$. Let $E \subset X$ be a measurable set such that $\mu(T^{-1}(E)\Delta E) = 0$. Consider a set $E_0 = \{x \in X | T^k(x) \in E \text{ for infinitely many } k \in \mathbb{N}\}$.

   (a) Show that $T^{-1}E_0 = E_0$.

   (b) Show that $E_0 \Delta E \subset \bigcup_{k=1}^{\infty} E\Delta T^{-k}(E)$.

   (c) Show that $E\Delta T^{-k}(E) \subset \bigcup_{i=0}^{k-1} T^{-i}(E)\Delta T^{-(i+1)}(E)$ for any $k \in \mathbb{N}$.

   (d) Using the previous items, show that $\mu(E_0\Delta E) = 0$.

   (e) Using the previous items, deduce that $\mu(E) = 0$ or $\mu(E) = 1$.

5. Let $X \subset \mathbb{R}^k$ and μ be a measure on X such that $\mu(X) = 1$. Consider a measure-preserving map $T\colon X \to X$. Assume that if $E \subset X$ is measurable and $\mu(T^{-1}(E)\Delta E) = 0$ then $\mu(E) = 0$ or $\mu(E) = 1$. Show that T is ergodic.

   Remark: Notice that the previous two problems show that T is ergodic if and only if we have that if $E \subset X$ is measurable and $\mu(T^{-1}(E)\Delta E) = 0$ then $\mu(E) = 0$ or $\mu(E) = 1$.

6. Let $\alpha \in \mathbb{Q}$. Consider $R_\alpha\colon S^1 \to S^1$ defined by $R_\alpha(x) = x + \alpha \pmod 1$. Prove that R_α is not ergodic with respect to the Lebesgues measure.