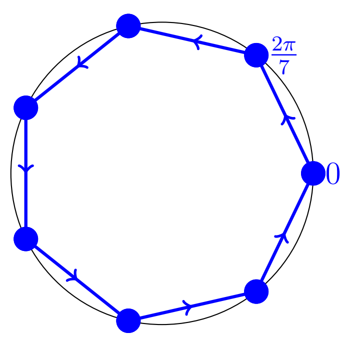
# Homework 6 - Solutions

MAT 351, Instructor: Alena Erchenko

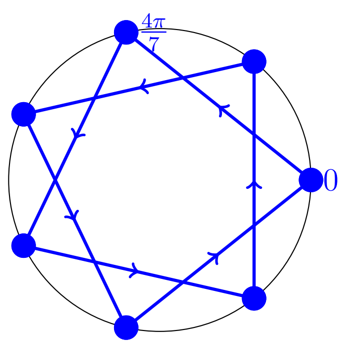
1. (a) *Solution.* We show that the given rotations are not topologically conjugate. Assume, for contradiction, that the rotation $R_{\frac{2}{5}}\colon S^1 \to S^1$ and $R_{\frac{1}{7}}\colon S^1 \to S^1$ are topologically conjugate. Then, there exists a homeomorphism $h\colon S^1 \to S^1$ such that $h \circ R_{\frac{2}{5}} = R_{\frac{1}{7}} \circ h$. Therefore, for any $n \in \mathbb{N}$ and $p \in S^1$ we have $h(R^n_{\frac{2}{5}}(p)) = R^n_{\frac{1}{7}}(h(p))$. Notice that for $R_{\frac{2}{5}}$ all points in $S^1$ are periodic with prime period 5 but for $R_{\frac{1}{7}}$ none of the points in $S^1$ are periodic with period 5 (actually, all points are periodic with prime period 7). Thus, we obtain contradiction that topological conjugacy $h$ exists as if it existed then $h(0) = h(R^5_{\frac{2}{5}}(0)) = R^5_{\frac{1}{7}}(h(0))$, i.e., $h(0)$ is periodic with period 5 for $R_{\frac{1}{7}}$. □

   (b) *Solution.* We show that $f$ and $g$ are not topologically conjugate. Assume, for contradiction, that they are topologically conjugate. Then, there exists a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ such that $h \circ g = f \circ h$. Therefore, for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have $h(g^n(x)) = f^n(h(x))$. We have $g'(x) = \frac{1}{3} \in (0, 1)$ for all $x \in \mathbb{R}$ so $g$ is contraction on $\mathbb{R}$. Also, notice that 0 is the unique fixed point of $g$ on $\mathbb{R}$. Thus, by the contraction principle, for any $x \in \mathbb{R}$ we have $g^n(x) \to 0$ as $n \to \infty$ (notice that 0 is the unique fixed point of $g$). Therefore, for any $x \in \mathbb{R}$ we have $\lim_{n\to\infty} h(g^n(x)) = h(0)$ (because $h$ is a continuous function) so $\lim_{n\to\infty} f^n(h(x)) = h(0)$ contradicting the fact that $\lim_{n\to\infty} f^n(h(h^{-1}(1))) = \lim_{n\to\infty} f^n(1) = \lim_{n\to\infty} 3^n = \infty$. □

2. *Solution.* (a)




   (b)



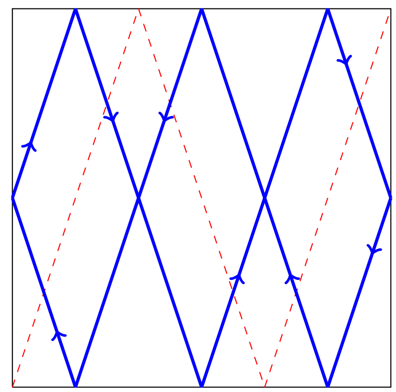
(c) Let the disk be given by $\{(x,y) \in \mathbb{R}^2 | x^2+y^2 \leq 1\}$. Notice that $\frac{1}{7\pi}$ is irrational so the orbit of $(s, \frac{1}{7})$ is dense in the annulus $\{(x,y) \in \mathbb{R}^2 | \cos\left(\frac{1}{7}\right) \leq x^2+y^2 \leq 1\}$.

□

3. *Solution.* Let the square be given in $\mathbb{R}^2$ by $\{(x,y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

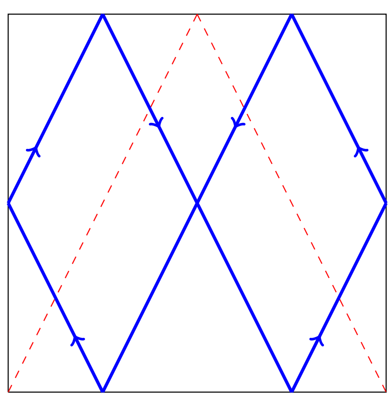
(a) Let $n \in \mathbb{N} \setminus \{1\}$ be even so $n = 2k$ for some $k \in \mathbb{N}$. Then, look at the orbit of the billiard flow starting at the left bottom vertex $(0,0)$ of the square with the direction given by vector $(1, 2k-1)$ which will terminate at the right top vertex $(1,1)$ of the square. Then, the orbit of any point on $\{(0,y) | 0 < y < 1\}$ with the direction $(1, 2k-1)$ will be periodic with period $2(2k-1)+2 = 4k = 2n$.

To obtain orbit with period 8, i.e., for $n = 4$, we have



$\frac{1}{3}$

(b) Let $n \in \mathbb{N} \setminus \{1\}$ be odd so $n = 2k+1$ for some $k \in \mathbb{N}$. Then, look at the orbit of the billiard flow starting at the left bottom vertex $(0,0)$ of the square with the direction given by vector $(1, 2k)$ which will terminate at the right bottom vertex $(1,0)$ of the square. Then, the orbit of any point on $\{(0,y) | 0 < y < 1\}$ with the direction $(1, 2k)$ will be periodic with period $2k+1+2k+1 = 2n$.



$\frac{1}{2}$

For $n = 3$ we have

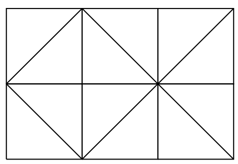
□

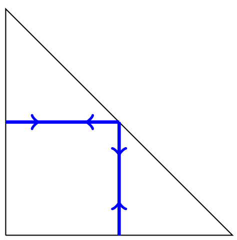
4. *Proof.* Let fix $n \in \mathbb{N}$. There are infinitely many orbits with prime period $2n$. If $n = 1$, we have any point $(s, \frac{\pi}{2})$ which is not a vertex of the square has prime period 2 where $s$ is the length parameter on the boundary of the square. If $n > 1$, then 1 and $n-1$ are relatively prime and $2 \cdot 1 + 2 \cdot (n-1) = 2n$. Let our square be given in $\mathbb{R}^2$ by $\{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

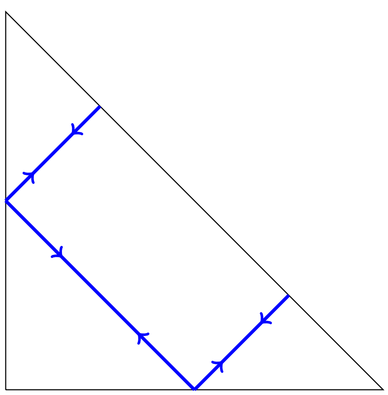
Then, for example, any point $(s, \arctan(n-1))$ with parameters $s$ corresponding to the points $\{(x,y) \in \mathbb{R}^2 | x = 0 \text{ and } 0 < y < 1\}$ is periodic with prime period $2n$. Notice that the line $y = y_0 + (n-1)x$ in $\mathbb{R}^2$ with $y_0 \in (0,1)$ doesn't pass through the points in the tiling corresponding to the vertices of the square. □

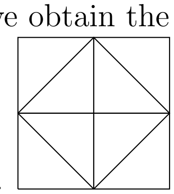
5. *Solution.* (a) There are no orbits with period 2 as such orbit should be orthogonal to two sides of the triangle but that would contradict the fact that sum of angles of a triangle in the plane is $\pi$.

   (b) Triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{4}$ has to have two sides that are not hypotenuse having the same length $L = \frac{\text{length of hypotenuse}}{\sqrt{2}}$ as $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$. If we reflect the triangle with respect to the hypotenuse, then we obtain obtain a square with sides of length $L$. Reflecting a square across a side corresponds to combination of reflections of the given triangle with respect to some sides. "Unfolding" of the square gives a tiling of the plane so "unfolding" of the given triangle gives a tiling of the plane.

   Sample of the tiling: 

   (c) 

   (d) 

   (e) We can get "unfolding" to the torus by doing the following reflection until we obtain the following pattern where the opposite parallel sides of the square are identified. 

   □

6. *Solution.* (a) $(0,0)$ is a periodic point of $f$ if and only if there exists $n \in \mathbb{N}$ such that $f^n(0,0) = (0,0)$, i.e., $(n\alpha, n\beta) = (0,0) \pmod 1$, meaning, $(n\alpha, n\beta) = (k,m)$ for some $k, m \in \mathbb{Z}$. Thus, $(0,0)$ is a periodic point of $f$ if and only if $\alpha$ and $\beta$ are rational numbers.

(b) No. Consider $\alpha$ being an irrational number and $\beta = 0$. Then, the orbit of $(0, 0)$ stays on the circle $\{(x, 0) | x \in S^1\}$ and fills it densely, so it is not periodic and is not dense in $\mathbb{T}^2$.

(c) Let $(x, y) \in \mathbb{T}^2$ and $d$ is the distance function on $\mathbb{T}^2$.

$(0, 0)$ has a dense orbit under $f$ in $\mathbb{T}^2$

$\Leftrightarrow$

For any $\varepsilon > 0$ for any $(a, b) \in \mathbb{T}^2$ there exists $N \in \mathbb{N}$ such that $d((a, b), f^N(0, 0)) < \varepsilon$

$\Leftrightarrow$

For any $\varepsilon > 0$ for any $(a, b) \in \mathbb{T}^2$ there exists $M \in \mathbb{N}$ such that $d((a-x, b-y), f^M(0, 0)) < \varepsilon$

$\Leftrightarrow$

For any $\varepsilon > 0$ for any $(a, b) \in \mathbb{T}^2$ there exists $M \in \mathbb{N}$ such that $d((a, b), f^M(x, y)) < \varepsilon$

$\Leftrightarrow$

$(x, y)$ has a dense orbit under $f$ in $\mathbb{T}^2$.

$\square$