Homework 6

MAT 351, Instructor: Alena Erchenko

1. For each given pair of dynamical systems, determine whether they are topologically conjugate or not. If they are topologically conjugate then give a formula for a topological conjugacy and prove that it is indeed a topological conjugacy. If not, explain why not.

   Hint: Recall definition of a topological conjugacy and Homework 5 Problem 5.

   (a) The rotations of the circle S^1 by $\frac{2}{5}$ and $\frac{1}{7}$.

   (b) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x$ and $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{3}x$.

2. Consider the billiard in the disc of radius 1. No explanations are required.

   (a) Sketch the orbit of $(s, \frac{\pi}{7})$ for some s.

   (b) Sketch the orbit of $(0, \frac{2\pi}{7})$ for some s.

   (c) What can you say about the orbit of $(s, \frac{1}{7})$ for any s? You can use the fact that π is irrational.

3. Consider the billiard in the unit square. Suggest a way to obtain an orbit with prime period $2n$ for $n \in \mathbb{N} \setminus \{1\}$ where

   (a) n is even;

   (b) n is odd.

   You don't need to prove that your method indeed gives a periodic orbit of the required period.

   Hint: Try to generalize the constructions for $n = 2$ and $n = 3$ in the lecture.

4. Consider the billiard in the unit square. Fix $n \in \mathbb{N}$. Explain if there are finitely (provide the number if it is the case) or infinitely many orbits with prime period $2n$.

5. Consider the billiard in a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{4}$ and $\frac{\pi}{4}$.

   (a) Explain why there are no orbit with period 2.

   (b) Can you obtain a tiling of the plane by "unfolding" construction? Draw the tiling or explain why it doesn't exist.

   (c) Draw an orbit of prime period 4 or explain why none exists.

   (d) Draw an orbit of prime period 6 or explain why none exists.

   (e) "Unfolding" to which surface can you obtain?

6. Let $(\alpha, \beta) \in \mathbb{R}^2$. Consider the translation $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ given by $f(x, y) = (x+\alpha, y+\beta) \pmod 1$.

(a) Continue the statement: $(0, 0)$ is a periodic point of f if and only if α and β .... Explain your answer.

(b) Is it true that if $(0, 0)$ is not periodic point for f, then its orbit is dense in $\mathbb{T}^2$? Explain your answer.

(c) Prove that $(0, 0)$ has a dense orbit under f in $\mathbb{T}^2$ if and only if any $(x, y) \in \mathbb{T}^2$ has a dense orbit under f in $\mathbb{T}^2$.