**Homework 7 - Solutions**

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1. (a) *Solution.* To find eigenvalues, we solve the equation

$$\lambda^2 - 2\lambda + 1 = 0, \qquad \text{i.e.} \quad (\lambda - 1)^2 = 0.$$

We obtain one eigenvalue $\lambda = 1$. Since the given matrix isn't $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We obtain that the given matrix is similar to $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. □

(b) *Solution.* To find eigenvalues, we solve the equation

$$\lambda^2 - 5\lambda + 6 = 0, \qquad \text{i.e.} \quad (\lambda - 2)(\lambda - 3) = 0.$$

We obtain two distinct eigenvalues $\lambda = 2$ and $\mu = 3$. We obtain that the given matrix is similar to $B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

□

(c) *Solution.* To find eigenvalues, we solve the equation

$$\lambda^2 - 6\lambda + 13 = 0.$$

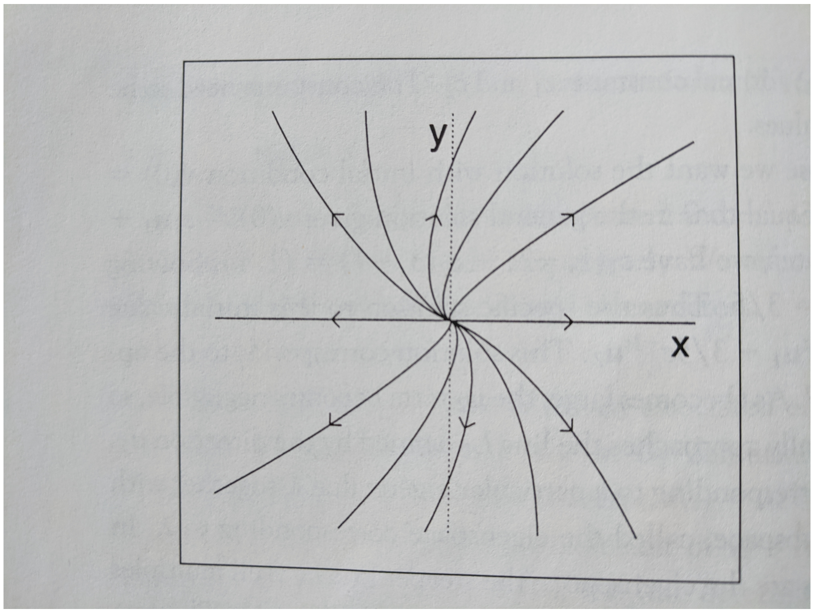
There are no real eigenvalues as $D = 6^2 - 4 \cdot 13 = -16 < 0$. The eigenvalues are complex $\lambda = 3 + 2i$ and $\mu = 3 - 2i$ so it is similar to a multiple of a rotation. We obtain that the given matrix is similar to $B = \sqrt{13}\begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} = r\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ where $r = \sqrt{13}$ and $\tan(\theta) = \frac{3}{2}$.

□

2. *Solution.* (a) The system can rewritten as $\dot{\bar{x}} = A\bar{x}$ where $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. We have that $A$ has one eigenvalue 3 with one eigenvector up to a scalar multiple. Thus, the zero-solution is a degenerate unstable node. The general solution is

$$\bar{x}(t) = e^{3t}(c_2 t + c_1)\bar{e}_1 + c_2 e^{3t}\bar{e}_2 \qquad \text{where} \qquad c_1, c_2 \in \mathbb{R}.$$

The other way to write is $x(t) = e^{3t}(c_2 t + c_1)$ and $y(t) = c_2 e^{3t}$ where $c_1, c_2 \in \mathbb{R}$.
The phase portrait is the following.



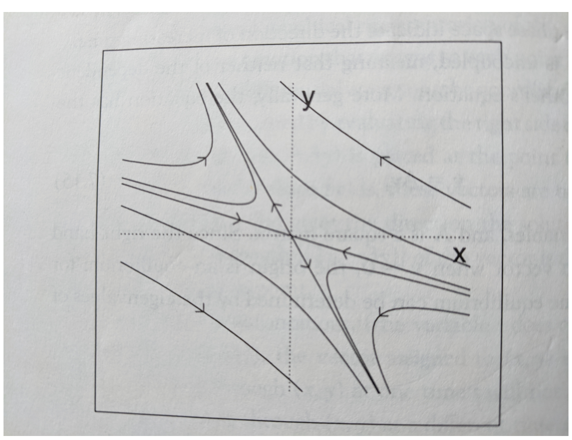

(b) The system can rewritten as $\dot{\bar{x}} = A\bar{x}$ where $A = \begin{pmatrix} -4 & -3 \\ 2 & 3 \end{pmatrix}$. To find eigenvalues, we solve the equation

$$\lambda^2 + \lambda - 6 = 0, \qquad \text{i.e.} \quad (\lambda+3)(\lambda-2) = 0.$$

We have that $A$ has two distinct eigenvalues $-3$ and $2$ with eigenvectors $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$,respectively. Thus, the zero-solution is a saddle. We have $\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} = C^{-1}AC$ where $C = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix}$.The general solution is

$$\bar{x}(t) = c_1 e^{-3t} C\bar{e}_1 + c_2 e^{2t} C\bar{e}_2 \qquad \text{where} \qquad c_1, c_2 \in \mathbb{R}.$$

The other way to write is $x(t) = 3c_1e^{-3t} + c_2e^{2t}$ and $y(t) = -c_1e^{-3t} - 2c_2e^{2t}$ where $c_1, c_2 \in \mathbb{R}$. The phase portrait is the following.




$\square$

3. *Solution.* (a) Since $A$ and $B$ are similar there exists an invertible matrix $C$ such that $B = C^{-1}AC$. Let $h(\bar{v}) = C\bar{v}$. Then, $h$ has inverse $h^{-1}(\bar{v}) = C^{-1}\bar{v}$. Also, $F \circ h = h \circ G$ as $CB = AC$. We have that $h$ is continuous as for $C$ there exists $r$ such that $d(C\bar{u}, C\bar{v}) \leq rd(\bar{u}, \bar{v})$ for all $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$. Thus, for any $\bar{u} \in \mathbb{R}^2$ and any $\varepsilon > 0$ there exists $\delta = \frac{\varepsilon}{r} > 0$ such that if $\bar{v} \in \mathbb{R}^2$ is such that $d(\bar{u}, \bar{v}) < \delta$ then $d(C\bar{u}, C\bar{v}) < \varepsilon$ so $h$ is continuous. Similarly, $h^{-1}$ is continuous.

Therefore, $F$ and $G$ are topologically conjugate and $h$ is a topological conjugacy.

(b) Let $h(x) = sign(x)|x|^\alpha$ for all $x \in \mathbb{R}$ where $\alpha$ is such that $b = a^\alpha$. We have that

$$\begin{aligned} h \circ f(x) = h(f(x)) = h(ax) &= sign(ax)|ax|^\alpha = sign(x)a^\alpha|x|^\alpha \\ &= sign(x)b|x|^\alpha \qquad \text{because} \quad a > 0 \quad \text{and} \quad b = a^\alpha \\ &= b\, sign(x)|x|^\alpha = g(h(x)) = g \circ h(x). \end{aligned}$$

Moreover, $h$ is invertible with inverse $h^{-1}(x) = sign(x)|x|^{1/\alpha}$ as

$$h(h^{-1}(x)) = sign\left(sign(x)|x|^{1/\alpha}\right)\left|sign(x)|x|^{1/\alpha}\right|^\alpha = sign(x)\left(|x|^{1/\alpha}\right)^\alpha = sign(x)|x| = x$$

and

$$h^{-1}(h(x)) = sign\left(sign(x)|x|^\alpha\right)\left|sign(x)|x|^\alpha\right|^{1/\alpha} = sign(x)|x| = x$$

for any $x \in \mathbb{R}$.

We show that $h$ is continuous. For any $x > 0$ there exists $\delta_1 > 0$ such that if $|x - y| < \delta_1$ then $y > 0$. Similarly, for $x < 0$ there exists $\delta_2 > 0$ such that if $|x - y| < \delta_2$ then $y > 0$. Let $x \neq 0$. Then, using the previous observations and the continuity of $|x|^\alpha$ on $\mathbb{R}$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then $sign(x) = sign(y)$ and

$$|h(x) - h(y)| = |sign(x)|x|^\alpha - sign(y)|y|^\alpha| = ||x|^\alpha - |y|^\alpha| < \varepsilon.$$

Let $x = 0$ then from continuity of $|x|^\alpha$ at $x = 0$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then

$$|h(0) - h(y)| = |sign(y)|y|^\alpha| = |y|^\alpha = ||0|^\alpha - |y|^\alpha| < \varepsilon.$$

Similarly, we can show that $h^{-1}$ is continuous.

Thus, $f$ and $g$ are topologically conjugate and $h$ is a topological conjugacy.

(c) Let $h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} sign(x)x^2 \\ y^3 \end{pmatrix}$ for any $x, y \in \mathbb{R}$.

We have that $h \circ F = G \circ h$ because for any $x, y \in \mathbb{R}$,

$$h(F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)) = h\left(\begin{pmatrix} \frac{x}{2} \\ \frac{y}{2} \end{pmatrix}\right) = \begin{pmatrix} sign(\frac{x}{2})\frac{x^2}{4} \\ \frac{y^3}{8} \end{pmatrix} = \begin{pmatrix} sign(x)\frac{x^2}{4} \\ \frac{y^3}{8} \end{pmatrix} = G(h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)).$$

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Assume that $d(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}) < \delta$. Then, $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ so $|x - a| < \delta$ and $|y - b| < \delta$. Using the continuity of functions $p(x) = sign(x)x^2$ and $r(x) = x^3$ on $\mathbb{R}$, for any $x, y \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $\delta$ such that

if $|x-a|<\delta$ then $|sign(x)x^2 - sign(a)a^2| < \frac{\varepsilon}{2}$
and
if $|y-b|<\delta$ then $|y^3-b^3|<\frac{\varepsilon}{2}$.
Then, for such choice of $\delta$ we have

$$d(h(\begin{pmatrix} x \\ y \end{pmatrix}), h(\begin{pmatrix} a \\ b \end{pmatrix})) = \sqrt{(sign(x)x^2 - sign(a)a^2)^2 + (y^3-b^3)^2} < \sqrt{\frac{\varepsilon^2}{2}} < \varepsilon.$$

Therefore, $h$ is continuous.

Notice that the inverse of $h$ is $h^{-1}(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} sign(x)|x|^{1/2} \\ y^{1/3} \end{pmatrix}$ which is also continuous by the argument similar to the continuity for $h$.

Therefore, $F$ and $G$ are topological conjugate and $h$ is a topological conjugacy.

□

4. *Solution.* (a) From the lecture we have that the linear system has the solution $\bar{x}(t) = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\mu t} \end{pmatrix}$ where $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \bar{x}(0)$. Thus,

$$f^t(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

so $f^1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ so $B = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix}$.

(b) From the lecture we have that the linear system has the solution $\bar{x}(t) = \begin{pmatrix} e^{\lambda t}(c_2 t + c_1) \\ c_2 e^{\lambda t} \end{pmatrix}$ where $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \bar{x}(0)$. Thus,

$$f^t(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

so $f^1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ so $B = \begin{pmatrix} e^{\lambda} & e^{\lambda} \\ 0 & e^{\lambda} \end{pmatrix}$.

(c) From the lecture we have that the linear system has the solution $\bar{x}(t) = \begin{pmatrix} c_1 \cos(rt) + c_2 \sin(rt) \\ c_1 \sin(rt) - c_2 \cos(rt) \end{pmatrix}$ where $\begin{pmatrix} c_1 \\ -c_2 \end{pmatrix} = \bar{x}(0)$. Thus,

$$f^t(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} \cos(rt) & -\sin(rt) \\ \sin(rt) & \cos(rt) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

so $f^1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ so $B = \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix}$.

□

5. *Solution.* (a) Let $t = Trace(A)$ and $d = \det(A)$. We have $d = \pm 1$ by the setting of the problem. Since $A$ has integer entries $t \in \mathbb{Z}$. The eigenvalues of $A$ are given by the formula $\frac{t \pm \sqrt{t^2-4d}}{2}$. The eigenvalues will be rational if and only if $t^2 - 4d = s^2$ for some integer $s$ (recall that $(t^2 - 4d) \in \mathbb{Z}$ as $t, d \in \mathbb{Z}$). Thus, $t^2 - s^2 = 4d = \pm 4$ so either $t, s$ are both even or $t, s$ are both odd.

<u>Case 1:</u> Assume $t, s$ are both odd so $t = 2l + 1$ and $s = 2m + 1$ for some $l, m \in \mathbb{Z}$. Then, $\pm 4 = t^2 - s^2 = (2l + 1)^2 - (2m + 1)^2$ so $\pm 1 = l^2 - m^2 + (l - m) = (l - m)(l + m + 1)$ so $l - m$ and $l + m + 1$ divide 1. Since the only integers that divide 1 are $\pm 1$, we obtain there are no integer solutions $l$ and $m$.

<u>Case 2:</u> Assume $t, s$ are both even so $t = 2l$ and $s = 2m$ for some $l, m \in \mathbb{Z}$. Then, $\pm 4 = t^2 - s^2 = 4l^2 - 4m^2$ so $\pm 1 = l^2 - m^2 = (l - m)(l + m)$ so $l - m$ and $l + m$ divide 1. Thus, the only possible solutions are ($l = \pm 1$ and $m = 0$) or ($l = 0$ and $m = \pm 1$) so ($t = \pm 2$ and $s = 0$) or ($t = 0$ and $s = \pm 2$). That implies that eigenvalues of $A$ can be only $\pm 1$.

Since $A$ is hyperbolic, we obtain that the eigenvalues of $A$ must be irrational.

(b) Let $l$ be a line passing through $(0, 0)$ spanned by an eigenvector of $A$. Notice that $l \neq \{(0, y) | y \in \mathbb{R}\}$ as, otherwise, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector so $A$ has form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ where $a, c, d \in \mathbb{Z}$ so $A$ has rational eigenvalues $a$ and $d$ which contradicts that $A$ has irrational eigenvalues (see the item 1). Therefore, we have $l = \{(x, y) \in \mathbb{R}^2 | y = \alpha x\}$ for some $\alpha \in \mathbb{R}$.

First, we prove that $l$ has a rational slope if and only if $l \cap \mathbb{Z}^2 \neq \{(0, 0)\}$.

If $l = \{(x, y) | y = \frac{p}{q} x\}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{Z} \setminus \{0\}$, then $(q, p) \in l$ so $(q, p) \in l \cap \mathbb{Z}^2$. Since $q \neq 0$, $l \cap \mathbb{Z}^2 \neq \{(0, 0)\}$. Conversely, if $(m, n) \in l \cap \mathbb{Z}^2 \setminus \{(0, 0)\}$, then $(m, n) \in l$and $(m, n) \neq (0, 0)$. In particular, $m \neq 0$ as otherwise $l = \{(0, y) | y \in \mathbb{R}\}$ which is not the case. Thus, we have $n = \alpha m$, i.e., $\alpha = \frac{n}{m} \in \mathbb{Q}$.

Assume that $l \cap \mathbb{Z}^2 \neq \{(0, 0)\}$. Without loss of generality, let the direction of $l$ be an eigenvector corresponding to the eigenvalue $\lambda$ such that $|\lambda| < 1$ (if not, replace $A$ with $A^{-1}$). Notice that $A(\mathbb{Z}^2) = \mathbb{Z}^2$ because $A$ has integer entries and invertible as $\det(A)$ is equal to the product of eigenvalues so $\det(A) \neq 0$. Since $A(l) = l$, we have that $A(l \cap \mathbb{Z}^2) = l \cap \mathbb{Z}^2$. Since $l \cap \mathbb{Z}^2 \neq \{(0, 0)\}$, there exists a point $(m, n) \neq (0, 0)$ in $l \cap \mathbb{Z}^2$ closest to $(0, 0)$. Then, $A(m, n) \in l \cap \mathbb{Z}^2$ and, moreover, $A(m, n) \neq (0, 0)$ as $A$ is invertible and $(m, n) \neq (0, 0)$, and $d(A(m, n), (0, 0)) = |\lambda| d((m, n), (0, 0))$ as $(m, n) \in l$ and $l$ is spanned by the eigenvector. Thus, since $|\lambda| < 1$, $d(A(m, n), (0, 0)) < d((m, n), (0, 0))$ and $A(m, n) \in l \cap \mathbb{Z}^2 \setminus \{(0, 0)\}$ contradicting the fact that $(m, n) \in l \cap \mathbb{Z}^2 \setminus \{(0, 0)\}$ was the closest to $(0, 0)$. Hence, $l \cap \mathbb{Z}^2 = \{(0, 0)\}$ so $l$ has an irrational slope.

$\square$