Homework 7

MAT 351, Instructor: Alena Erchenko

1. For each of the following matrices A, find a "model" matrix B such that A is similar to B. If B is a scalar multiple of a rotation matrix then determine its parameters θ and r. Show your work.

   (a) $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$;

   (b) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$;

   (c) $\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$.

2. Find the solutions of the following linear systems, sketch the phase portraits, and determine the type of the zero-solution.

   (a)
   $$\begin{aligned} \dot{x} &= 3x + y \\ \dot{y} &= 3y. \end{aligned}$$

   (b)
   $$\begin{aligned} \dot{x} &= -4x - 3y \\ \dot{y} &= 2x + 3y. \end{aligned}$$

3. For each given pair of dynamical systems, show that they are topologically conjugate. Explain your answer.

   (a) $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(\bar{v}) = A\bar{v}$ and $G(\bar{v}) = B\bar{v}$ where A and B are similar 2×2 matrices.

   Hint: Recall that the distance on $\mathbb{R}^2$ is defined in the following way $d(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$. You can use the fact that for any 2×2 matrix D there exists a constant r such that $d(D\bar{u}, D\bar{v}) \leq r d(\bar{u}, \bar{v})$ for all $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$.

   (b) $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax$ and $g(x) = bx$, where $0 < a < b < 1$.

   Hint: Consider $h\colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = sign(x)|x|^\alpha$ where α is such that $b = a^\alpha$ and $sign(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$. You can use the fact that $|x|^l$ where $l > 0$ is a continuous function on $\mathbb{R}$.

(c) $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(\bar{v}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \bar{v}$ and $G(\bar{v}) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{8} \end{pmatrix} \bar{v}$.

<u>Hint:</u> Use the previous item.

4. Consider a flow $f^t\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by the linear system $\dot{\bar{x}} = A\bar{x}$ where A is one of the following "model" matrices. Determine the matrix B for the time-one-map $f^1\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f^1(\bar{x}) = B\bar{x}$.

   (a) $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $\lambda, \mu \in \mathbb{R}$;

   (b) $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ where $\lambda \in \mathbb{R}$;

   (c) $\begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix}$ where $r > 0$.

5. We say that a 2×2 matrix A is hyperbolic if it has two distinct eigenvalues λ and μ such that $0 < |\mu| < 1 < |\lambda|$. Let A be a 2×2 hyperbolic matrix with integer entries and determinant ± 1. Show that

   (a) The eigenvalues of A are irrational;

   (b) (optional) The eigenvectors of A have irrational slopes.

   <u>Hint:</u> Let l be a line spanned by an eigenvector of A passing through $(0,0)$.

   i. Prove that that a line l has rational slope if and only if $l \cap \mathbb{Z}^2 \neq \{(0,0)\}$.

   ii. Prove that for a line l we have $A(l \cap \mathbb{Z}^2) = l \cap \mathbb{Z}^2$.