Homework 8 - Solutions

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1. *Solution.* We have $\dot{\sin}(t) = \cos(t)$ and $\dot{\cos}(t) = -\sin(t) = -\sin(t) + (1 - \sin^2(t) - \cos^2(t))\cos(t)$, using the trigonometric identity $\sin^2(t) + \cos^2(t) = 1$. Thus, $x(t) = \sin(t)$ and $y(t) = \cos(t)$ is a solution of the system. Geometrically, the trajectory starts at the point $(0, 1)$ at time $t = 0$ and moves clockwise around the unit circle centered at $(0, 0)$ as $x^2(t) + y^2(t) = 1$ for all $t \in \mathbb{R}$. □

2. (a) *Solution.* To find equilibrium states, we need to solve the system of equations $x - y = 0, \quad x^2 - 4 = 0$. The second equation gives us that $x = \pm 2$ and the first equation gives us that $y = x$. Thus, the equilibrium states are $(2, 2)$ and $(-2, 2)$.

   The Jacobian matrix at (x, y) is $A(x, y) = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$.

   We have $A(2, 2) = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$ so the characteristic equation is $\lambda^2 - \lambda + 4 = 0$ so it does not have real eigenvalue, but it has complex eigenvalues $\lambda = \frac{1 \pm i\sqrt{15}}{2}$. Since $Re(\lambda) = \frac{1}{2} > 0$ and the eigenvalues are complex, we have that $(2, 2)$ is an unstable spiral.

   We have $A(-2, -2) = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}$ so the characteristic equation is $\lambda^2 - \lambda - 4 = 0$ so the eigenvalues are $\lambda = \frac{1+\sqrt{17}}{2}$ and $\mu = \frac{1-\sqrt{17}}{2}$. Since $\lambda > 0$ and $\mu < 0$, we have that $(-2, -2)$ is a saddle point. □

   (b) *Solution.* To find equilibrium states, we need to solve the system of equations $\sin(y) = 0, \quad \cos(x) = 0$. We obtain $y = \pi k$ where $k \in \mathbb{Z}$ and $x = \frac{\pi}{2} + \pi m$ where $m \in \mathbb{Z}$. Thus, the equilibrium states are $(\frac{\pi}{2} + \pi m, \pi k)$ where $m, k \in \mathbb{Z}$.

   The Jacobian matrix at (x, y) is $A(x, y) = \begin{pmatrix} 0 & \cos(y) \\ -\sin(x) & 0 \end{pmatrix}$.

   We have $A(\frac{\pi}{2} + \pi m, \pi k) = \begin{pmatrix} 0 & (-1)^k \\ (-1)^{m+1} & 0 \end{pmatrix}$ so the characteristic equation is $\lambda^2 + (-1)^{k+m+1} = 0$ so if $k + m = 2l$ for some $l \in \mathbb{Z}$ then the eigenvalues are ± 1 and if $k + m = 2l + 1$ for some $l \in \mathbb{Z}$ then there are no real eigenvalues and the complex eigenvalues are $\pm i$. Thus, the equilibrium states $(\frac{\pi}{2} + \pi m, \pi k)$ where $m, k \in \mathbb{Z}$ and $m + k$ is even are saddle points, and the other equilibrium states we cannot classify as the linearized system predicts center which is a "borderline case". □

   (c) *Solution.* To find equilibrium states, we need to solve the system of equations $xy - 1 = 0, \quad x - y^3 = 0$. The second equation gives us that $x = y^3$ so the first equation gives us that $y^4 = 1$ so $y = \pm 1$ and $x = y^3$. Thus, the equilibrium states are $(1, 1)$ and $(-1, -1)$.

   The Jacobian matrix at (x, y) is $A(x, y) = \begin{pmatrix} y & x \\ 1 & -3y^2 \end{pmatrix}$.

We have $A(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$ so the characteristic equation is $\lambda^2 + 2\lambda - 4 = 0$ so the eigenvalues are $1 \pm \sqrt{5}$. Since the eigenvalues are distinct amd have different signs, we have that $(1,1)$ is a saddle point.

We have $A(-1,-1) = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$ so the characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$ so $(\lambda+2)^2 = 0$ so there is one real eigenvalue -2. Since there is only one eigenvalue and the matrix is not diagonal, we cannot classify $(-1,-1)$ from the linear system we obtain that it corresponds to degenerate stable node which is a "borderline case". □

3. *Solution.* (a) To find equilibrium states, we need to solve the system of equations $y^3 - 4x = 0, \quad y^3 - y - 3x = 0$. The first equation gives us that $x = \frac{1}{4}y^3$ so the second equation gives us that $y^3 - 4y = 0$ so $y(y-2)(y+2) = 0$ so $y = 0$ or $y = 2$ or $y = -2$. Thus, the equilibrium states are $(0,0)$, $(2,2)$ and $(-2,-2)$.

   The Jacobian matrix at (x,y) is $A(x,y) = \begin{pmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{pmatrix}$.

   We have $A(0,0) = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix}$ so the eigenvalues are -4 and -1. Since the eigenvalues are distinct negative numbers, we have that $(0,0)$ is a stable node.

   We have $A(2,2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix}$ so the characteristic equation is $\lambda^2 - 7\lambda - 8 = 0$, i.e., $(\lambda - 8)(\lambda + 1) = 0$ the eigenvalues are 8 and -1. Since the eigenvalues are of different sign, we have that $(2,2)$ is a saddle point.

   We have $A(-2,-2) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} = A(2,2)$ so we have that $(-2,-2)$ is a saddle point.

   (b) $\frac{d}{dt}(x-y) = y^3 - 4x - (y^3 - y - 3x) = y - x = 0$ at points with $x = y$ so $x - y = 0$, i.e., $x = y$, is an invariant line.

   (c) $\frac{d}{dt}(x-y) = -(x-y)$ so $x(t) - y(t) = Ce^{-t}$ for some constant $C \in \mathbb{R}$ where $C = x(0) - y(0)$. Thus, if $x(0) \neq y(0)$ so we are not on the line $x = y$, then $\lim_{t\to\infty} |x(t) - y(t)| = \lim_{t\to\infty} |C|e^{-t} = 0$.

   (d) To sketch the phase portrait, we would like to find eigenvectors of $A(2,2) = A(-2,-2)$. An eigenvector for 8 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. An eigenvector for -1 is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. We also would like to find eigenvectors of $A(0,0)$. An eigenvector for -4 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and an eigenvector for -1 is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   See the picture of the phase portrait at the end of the homework solutions.

□

4. *Solution.* The circle rotations are not structurally stable. Consider $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha \pmod 1$ for $x \in S^1$. If $\alpha \in \mathbb{Q}$, then all orbits of R_α are periodic and if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then all orbits are dense. For any $\varepsilon > 0$ there exists $\beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\gamma \in \mathbb{Q}$ such that $|\beta - \alpha| < \varepsilon$ and $|\gamma - \alpha| < \varepsilon$. Thus, for any α we can find rotation on S^1 arbitrarily C^1-close to R_α with the different orbit behavior (all orbits periodic vs all orbits dense) so they cannot be topologically conjugate. Therefore, R_α is not structurally stable. □

5. *Proof.* Let $f\colon X \to X$ be a continuous map. Assume $x \in X$ is recurrent for f so $x = \lim_{k\to\infty} f^{n_k}(x)$ for some subsequence (n_k) such that $n_k \to \infty$ as $k \to \infty$. Then, $f(x) = f(\lim_{k\to\infty} f^{n_k}(x)) = \lim_{k\to\infty} f(f^{n_k}(x))$ as f is a continuous function. Thus, $f(x) = \lim_{k\to\infty} f^{n_k+1}(x)$ so $f(x)$ is recurrent for f. □

6. *Proof.* Let $X \subset \mathbb{R}$ be a closed interval and $f\colon X \to X$ is a λ-contraction (where $0 < \lambda < 1$). Since f is a contraction, by the Contraction principle, there exists a unique $c \in X$ such that $f(c) = c$ and for every $x \in X$ and $n \in \mathbb{N}$ we have $d(f^n(x), c) \le \lambda^n d(x, c)$.

   Let $x \in X$ be such that $f(x) \neq x$, i.e., $x \neq c$. Let $\varepsilon = \frac{(1-\lambda)d(x,c)}{2} > 0$. By the triangle inequality $d(x, c) \le d(x, f^n(x)) + d(f^n(x), c)$. Thus, for any $n \in \mathbb{N}$ we have

$$d(f^n(x), x) \ge d(x, c) - d(f^n(x), c) \ge d(x, c) - \lambda^n d(x, c) = (1 - \lambda^n)d(x, c) \ge (1 - \lambda)d(x, c) > \varepsilon.$$

   Therefore, no point in X, except for the fixed point, is recurrent. □

<u>The phase portrait for Problem 3d.</u>