Homework 8

MAT 351, Instructor: Alena Erchenko

1. Consider the system $\dot{x} = y$, $\dot{y} = -x + (1 - x^2 - y^2)y$. Show that $x(t) = \sin(t), \quad y(t) = \cos(t)$ is a solution of the system and describe geometrically the trajectory.

2. For each of the following systems, find the equilibrium states and classify them (if possible from the linearization).

   (a) $\dot{x} = x - y, \quad \dot{y} = x^2 - 4;$

   (b) $\dot{x} = \sin(y), \quad \dot{y} = \cos(x);$

   (c) $\dot{x} = xy - 1, \dot{y} = x - y^3.$

3. Consider the system $\dot{x} = y^3 - 4x, \quad \dot{y} = y^3 - y - 3x.$

   (a) Find all the equilibrium states and classify them;

   (b) Show that the line $x = y$ is invariant, i.e., any trajectory that starts on it stays on it;

   (c) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all other trajectories;
   Hint: Form a differential equation for $x - y$.

   (d) Sketch the phase portrait.

4. Are circle rotations structurally stable? Justify your answer.

5. Let $X \subset \mathbb{R}^k$. Consider a continuous map $f\colon X \to X$. Prove that if a point $x \in X$ is recurrent, then $f(x)$ is also recurrent.

   Remark: It implies that the set R of recurrent points satisfies $f(R) \subset R$.

6. Let $X \subset \mathbb{R}$ be a closed interval with the distance d coming from $\mathbb{R}$. Consider a contraction $f\colon X \to X$. Prove that no point in X, except for the fixed point, is recurrent, that is, show that if $f(x) \neq x$, then there is $\varepsilon > 0$ such that there is no $n \in \mathbb{N}$ with $d(f^n(x), x) < \varepsilon$.