Homework 9

MAT 351, Instructor: Alena Erchenko

1. Prove that the middle-thirds Cantor set has Lebesgue measure zero. Make sure to explain why the middle-thirds Cantor set is measurable using the description of the measurable sets in $\mathbb{R}$.

2. For any $a \in [0, 1)$, give an example of a set in $[0, 1]$ that has Lebesgue measure a, but does not contain any open intervals. Justify your answer.

3. Prove that the set $A \subset \mathbb{R}$ is closed if and only if its complement in $\mathbb{R}$ (i.e., a set $\mathbb{R} \setminus A$) is open.

4. Does there exists a closed set in $[0, 1]$ that has Lebesgue measure 1, but does not contain any open intervals? Justify your answer.

   Hint: Use the previous problem.

5. Let μ be the k-dimensional Lebesgue measure. Consider a set $X \subset \mathbb{R}^k$ of finite measure. Let $f \colon X \to X$ be a measure-preserving map. Consider a set $A \subset X$ such that $\mu(A) > 0$. Prove that there exists $n \in \mathbb{N}$ such that $\mu(A \cap f^{-n}(A)) > 0$, that is, the set $\{a \in A | f^n(a) \in A\}$ has positive measure.

   Hint: Use the Poincaré Recurrence Theorem.

6. Let $X = \mathbb{R}^2$ and let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$f(x, y) = (x + y, y) \qquad \text{given by the matrix} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

   Show that the conclusion of the Poincaré Recurrence Theorem fails for f.