Homework 10 - Solutions

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1. *Solution.* (a) Since $B_i \cap B_j = \emptyset$ if $i \neq j$, we have that if $x \in B_i$ then $x \notin B_j$ for all $j \neq i$. Therefore, $a_i \geq 0$ for all $i \in \{1, 2, \ldots, n\}$ because $s(x) \geq 0$ for all $x \in X$.
By definition, we have

$$\int_X s(x) d\mu(x) = \sum_{k=1}^{n} a_k \mu(B_k).$$

Thus, $\int_X s(x) d\mu(x) \geq 0$ because $a_k \geq 0$ and $\mu(B_k) \geq 0$ for all $k \in \{1, 2, \ldots, n\}$.

(b) Let $s(x) = \sum\limits_{k=1}^{n} a_k \chi_{A_k}(x)$ where $a_1, \ldots, a_n \in \mathbb{R}$ and $A_1, \ldots, A_n$ are measurable and $h(x) = \sum\limits_{k=1}^{m} b_k \chi_{B_k}(x)$ where $b_1, \ldots, b_m \in \mathbb{R}$ and $B_1, \ldots, B_m$ are measurable.
Then, $as(x)+bh(x) = \sum\limits_{k=1}^{n} aa_k \chi_{A_k}(x) + \sum\limits_{k=1}^{m} bb_k \chi_{B_k}(x)$ where $aa_1, \ldots, aa_n \in \mathbb{R}$ and $bb_1, \ldots, bb_m \in \mathbb{R}$ so it is a simple function . Thus,

$$\begin{aligned}
\int_X as(x) + bh(x) d\mu(x) &= \sum_{k=1}^{n} aa_k \mu(A_k) + \sum_{k=1}^{m} bb_k \mu(B_k) \\
&= a \sum_{k=1}^{n} a_k \mu(A_k) + b \sum_{k=1}^{m} b_k \mu(B_k) \\
&= a \int_X s(x) d\mu(x) + b \int_X h(x) d\mu(x).
\end{aligned}$$

(c) Let $f \colon X \to \mathbb{R}$ be a measurable integrable function. Define $f_+(x) = \begin{cases} f(x) & \text{if} \quad f(x) > 0 \\ 0 & \text{otherwise} \end{cases}$ and $f_-(x) = \begin{cases} -f(x) & \text{if} \quad f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$. Then, $|f(x)| = f_+(x) + f_-(x)$ for all $x \in X$. Also, since $f_+(x) \geq 0$ and $f_-(x) \geq 0$ for all $x \in X$, by the remark in the item (a), we have $\int_X f_+(x) d\mu(x) \geq 0$ and $\int_X f_-(x) d\mu(x) \geq 0$.

Therefore, using the remark from the item (b), we obtain

$$\begin{aligned}\left|\int_X f(x)d\mu(x)\right| &= \left|\int_X f_+(x)d\mu(x) - \int_X f_-(x)d\mu(x)\right| \\ &\leq \left|\int_X f_+(x)d\mu(x)\right| + \left|\int_X f_-(x)d\mu(x)\right| \\ &= \int_X f_+(x)d\mu(x) + \int_X f_-(x)d\mu(x) \\ &= \int_X (f_+(x) + f_-(x))d\mu(x) \\ &= \int_X |f(x)|d\mu(x).\end{aligned}$$

□

2. *Solution.* Let $A \subset X$ be a measurable set such that $T^{-1}(A) = A$. Then, $\chi_A \colon X \to \mathbb{R}$ is a measurable function and $\chi_A(T(x)) = \chi_{T^{-1}A}(x) = \chi_A(x)$ for all $x \in X$, i.e., $\chi_A \circ T = \chi_A$ everywhere so χ_A is constant almost everywhere. In particular, $\chi_A(x) = 1$ almost everywhere or $\chi_A(x) = 0$ almost everywhere, so $\mu(A) = 1$ or $\mu(A) = 0$. □

3. *Solution.* Let $A_1, A_2, A_3 \subset X$ be measurable sets.

   First, we show that $\mu(A_i \Delta A_j) = \int_X |\chi_{A_i}(x) - \chi_{A_j}(x)|d\mu(x)$. We have $A_i \Delta A_j = (A_i \setminus A_j) \cup (A_j \setminus A_i)$.

   We have that $\chi_{A_i \Delta A_j}(x)$ is equal to 1 if ($x \in A_i$ and $x \notin A_j$) or ($x \in A_j$ and $x \notin A_i$) and is equal to 0 otherwise. Thus, $\chi_{A_i \Delta A_j}(x) = |\chi_{A_i}(x) - \chi_{A_j}(x)|$ for all $x \in X$, because if $x \in X$ then ($x \in A_i$ and $x \notin A_j$) or ($x \in A_j$ and $x \notin A_i$) or ($x \in A_i$ and $x \in A_j$) or ($x \notin A_i$ and $x \notin A_j$) so we can check the values of the function.

   Hence,

$$\mu(A_i \Delta A_j) = \int_X \chi_{A_i \Delta A_j}(x)d\mu(x) = \int_X |\chi_{A_i}(x) - \chi_{A_j}(x)|d\mu(x).$$

   Furthermore, using the above, we obtain

$$\begin{aligned}\mu(A_1 \Delta A_3) &= \int_X |\chi_{A_1}(x) - \chi_{A_3}(x)|d\mu(x) \\ &= \int_X |(\chi_{A_1}(x) - \chi_{A_2}(x)) + (\chi_{A_2}(x) - \chi_{A_3}(x))|d\mu(x) \\ &\leq \int_X |\chi_{A_1}(x) - \chi_{A_2}(x)|d\mu(x) + \int_X |\chi_{A_2}(x) - \chi_{A_3}(x)|d\mu(x) \qquad \text{using Problem 1} \\ &= \mu(A_1 \Delta A_2) + \mu(A_2 \Delta A_3).\end{aligned}$$

   Hence, $\mu(A_1 \Delta A_3) \leq \mu(A_1 \Delta A_2) + \mu(A_2 \Delta A_3)$. □

4. *Solution.* Let $E \subset X$ be a measurable set such that $\mu(T^{-1}E \Delta E) = 0$.

(a) We have $x \in T^{-1}E_0$ if and only if $T(x) \in E_0$ if only if $T^k(T(x)) = T^{k+1}(x) \in E$ for infinitely many $k \in \mathbb{N}$ if and only if $T^m(x) \in E$ for infinitely many $m \in \mathbb{N}$ if and only if $x \in E_0$. Hence, $T^{-1}(E_0) = E_0$.

(b) Notice that

  i. if $x \in E_0 \setminus E$, then there exists some $k \in \mathbb{N}$ such that $x \in T^{-k}(E) \setminus E$ so $x \in E\Delta T^{-k}(E)$ for some $k \in \mathbb{N}$;

  ii. if $x \in E \setminus E_0$, then there exists some $k \in \mathbb{N}$ such that $x \notin T^{-k}(E)$ so $x \in E \setminus T^{-k}(E)$ so $x \in E\Delta T^{-k}(E)$ for some $k \in \mathbb{N}$.

Thus, if $x \in E_0 \Delta E$, then $x \in E\Delta T^{-k}(E)$ for some $k \in \mathbb{N}$ so

$$E_0\Delta E \subset \bigcup_{k=1}^{\infty} E\Delta T^{-k}(E).$$

(c) We prove that $E\Delta T^{-k}(E) \subset \bigcup_{i=0}^{k-1} T^{-i}(E)\Delta T^{-(i+1)}(E)$ for any $k \in \mathbb{N}$ by induction. The base case $k = 1$ is obvious as $\bigcup_{i=0}^{k-1} T^{-i}(E)\Delta T^{-(i+1)}(E) = E\Delta T^{-1}(E) = E\Delta T^{-k}(E)$ for $k = 1$. Assume we know the statement for k and want to show for $k + 1$.
We have $x \in E\Delta T^{-(k+1)}(E)$ so ($x \in E$ and $x \notin T^{-(k+1)}(E)$) or ($x \notin E$ and $x \in T^{-(k+1)}(E)$). Since $x \in X$, we have $x \in T^{-k}(E)$ or $x \notin T^{-k}(E)$. Considering all the possibilities, we obtain that $x \in T^{-k}(E)\Delta T^{-(k+1)}(E)$ or $x \in E\Delta T^{-k}(E)$, in particular, by the induction hypothesis, $x \in \bigcup_{i=0}^{k-1} T^{-i}(E)\Delta T^{-(i+1)}(E)$. Thus, $x \in \bigcup_{i=0}^{k} T^{-i}(E)\Delta T^{-(i+1)}(E)$ so $E\Delta T^{-(k+1)}(E) \subset \bigcup_{i=0}^{k} T^{-i}(E)\Delta T^{-(i+1)}(E)$ and we obtain the inductive step.
Hence, we proved the statement.

(d) Using item (c) and the properties of the measure that we discussed in class, we obtain for all $k \in \mathbb{N}$

$$\begin{aligned}\mu(E\Delta T^{-k}(E)) &\le \sum_{i=0}^{k-1} \mu(T^{-i}(E)\Delta T^{-(i+1)}(E)) = \sum_{i=0}^{k-1} \mu(T^{-i}\left(E\Delta T^{-1}(E)\right)) \\ &= k\mu(E\Delta T^{-1}(E)) = 0\end{aligned}$$

as T preserves μ and $\mu(E\Delta T^{-1}E) = 0$.
Thus, $\mu(E\Delta T^{-k}(E)) = 0$ for all $k \in \mathbb{N}$.
Using item (b) and the above, we obtain that

$$\mu(E_0\Delta E) \le \sum_{k=1}^{\infty} \mu(E\Delta T^{-k}(E)) = 0.$$

Thus, $\mu(E_0\Delta E) = 0$.

(e) Since $\mu(E_0\Delta E) = 0$, we have that $\mu(E_0 \setminus E) + \mu(E \setminus E_0) = 0$ so $\mu(E_0 \setminus E) = \mu(E \setminus E_0) = 0$. Thus, since $E = (E \setminus E_0) \cup (E \cap E_0)$ and $E_0 = (E_0 \setminus E) \cup (E_0 \cap E)$, we obtain that $\mu(E) = \mu(E_0)$. By item (a), $T^{-1}E_0 = E_0$ so $\mu(E_0) = 0$ or $\mu(E_0) = 1$ as T is ergodic. Therefore, $\mu(E) = 0$ or $\mu(E) = 1$.

□

5. *Solution.* Let $A \subset X$ be measurable and $T^{-1}(A) = A$. Then, $T^{-1}(A)\Delta A = \emptyset$ so $\mu(T^{-1}(A)\Delta A) = \mu(\emptyset) = 0$. Thus, $\mu(A) = 0$ or $\mu(A) = 1$ by the assumption. Therefore, T is ergodic by definition. □

6. *Solution.* Let $\alpha \in \mathbb{Q}$ so $\alpha = \frac{p}{q}$ where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and p and q are coprime.

   If $q = 1$, then $\alpha \in \mathbb{Z}$ so $R_\alpha(x) = x$ for all $x \in S^1$. Thus, $A = [0, \frac{1}{2}]$ is a measurable such that $R_\alpha^{-1}(A) = A$ and $0 < \mu(A) = \frac{1}{2} < 1$. Thus, R_α is not ergodic.

   We can assume that $0 < \frac{p}{q} < 1$ as if $\frac{p}{q} > 1$ then $\frac{p}{q} = n + \frac{\tilde{p}}{q}$ where $n \in \mathbb{N}$ and $\tilde{p}$ and q are coprime and $R_{\frac{p}{q}} = R_{\frac{\tilde{p}}{q}}$. If $\frac{p}{q} < 0$, then we can repeat a similar construction as below.

   Let $0 < \frac{p}{q} < 1$. Define $A = \bigcup_{n=0}^{q-1} [n \cdot \frac{p}{q}, n \cdot \frac{p}{q} + \frac{1}{2q}]$. Then, $R_{\frac{p}{q}}^{-1}(A) = A$ and $0 < \mu(A) = q \cdot \frac{1}{2q} = \frac{1}{2} < 1$. Thus, $R_{\frac{p}{q}}$ is not ergodic. □