Homework 9 - Solutions

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1. *Solution.* If $A \subset [0,1]$, we use the following notation for the sets $\frac{A}{3} = \{\frac{x}{3} | x \in A\}$ and $\frac{2}{3} + A = \{\frac{2}{3} + x | x \in A\}$.

   Let $A_1 = (\frac{1}{3}, \frac{2}{3})$ which is measurable as it is an open interval. Let $A_2 = (\frac{1}{3^2}, \frac{2}{3^2}) \cup (\frac{2}{3} + \frac{1}{3^2}, \frac{2}{3} + \frac{2}{3^2})$ which is measurable as it is a union of two measurable sets that are open intervals. We define $A_k = \frac{A_{k-1}}{3} \cup \left(\frac{2}{3} + \frac{A_{k-1}}{3}\right)$ for $k \geq 2$. If A_{k-1} is a union of open intervals (so it is measurable) so $\frac{A_{k-1}}{3}$ and $\left(\frac{2}{3} + \frac{A_{k-1}}{3}\right)$ are the union of open intervals so they are measurable. Since A_k is the union of those sets, we obtain that A_k is the union of open intervals so it is measurable. Let $A = \bigcup_{n=1}^{\infty} A_n$ then it is a measurable set as it is a countable union of measurable sets. Thus, the middle-thirds Cantor set $C = [0,1] \setminus A$ is a measurable set as $[0,1]$ is a closed interval so measurable and A is measurable.

   Denote the Lebesgue measure on $\mathbb{R}$ by μ. We have

$$\begin{aligned}
\mu(A_k) &= \mu\left(\frac{A_{k-1}}{3} \cup \left(\frac{2}{3} + \frac{A_{k-1}}{3}\right)\right) \\
&= \mu\left(\frac{A_{k-1}}{3}\right) + \mu\left(\left(\frac{2}{3} + \frac{A_{k-1}}{3}\right)\right) \quad \text{as } \frac{A_{k-1}}{3} \cap \left(\frac{2}{3} + \frac{A_{k-1}}{3}\right) = \emptyset \\
&= \frac{2}{3}\mu(A_{k-1}) \quad \text{using the Lebesgue measure of intervals.}
\end{aligned}$$

   Thus, $\mu(A_k) = \left(\frac{2}{3}\right)^{k-1} \mu(A_1) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}$.

   Notice that $A_j \cap A_i = \emptyset$ for $i \neq j$ so

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n) = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}} = 1.$$

   Therefore, $\mu(C) = \mu([0,1]) - \mu(A) = 1 - 1 = 0.$ □

2. *Solution.* Let $a \in [0,1)$. Consider $b = \frac{1-a}{1+2(1-a)} = \frac{1-a}{3-2a} \in (0, \frac{1}{3}]$. Consider the set C that is obtained by removing from the middle each closed interval starting from $[0,1]$ an open interval of length b times the length of the closed interval. Notice that by the construction, the set won't contain any open intervals: Let $A_0 = [0,1]$, $A_1 = [0, \frac{1}{2} - \frac{b}{2}] \cup [\frac{1}{2} + \frac{b}{2}, 1]$ and so on. Then, $C = \bigcap_{k=0}^{\infty} A_k$, in particular, C is measurable. Moreover, A_k contains 2^k maximal closed intervals of length at most $\frac{1}{2^k} \to 0$ as $k \to \infty$.

The other way to describe C is the following: C is equal to $[0, 1]$ without a countable union of the open intervals which is also shows that C is a measurable set. On the k-th step we remove a set of Lebesgue measure $(2b)^{k-1}b$. Thus, the resulting set has measure equal to $1 - b\sum_{n=1}^{\infty}(2b)^n = 1 - \frac{b}{1-2b} = a$. □

3. *Solution.* Let $A^c = \mathbb{R} \setminus A$.

   ⇒: Let $A \subset \mathbb{R}$ be closed, i.e., A contains all its limit points. Consider $x \in A^c$ so $x \notin A$. In particular, there exists $r > 0$ such that $B(x, r) \subset A^c$, where $B(x, r)$ is an open ball centered at x of radius r. Since this is the case for any $x \in A^c$, we have A^c is open by definition.

   ⇐: Assume A^c is open, so for any $a \in A^c$ there exists $r > 0$ such that $B(a, r) \subset A^c$. Let x be a limit point of A, so for any $r > 0$ there exists $y \in B(x, r)$ such that $y \in A$. Thus, $x \notin A^c$ as there is no $r > 0$ such that $B(x, r) \subset A^c$. Therefore, $x \in A$. Hence, A is closed. □

4. *Solution.* No.

   Assume, for contradiction, that there exists a closed set C in $[0, 1]$ that has Lebesgue measure 1, but does not contain any open intervals. Then, by the previous problem $\mathbb{R} \setminus C$ is open. In particular, the set $B = [0, 1] \cap (\mathbb{R} \setminus C)$ contains an open interval, so $\mu(B) > 0$ contradicting the fact that $\mu(B) = \mu([0, 1]) - \mu(C) = 1 - 1 = 0$, where μ is the Lebesgue measure. □

5. *Solution.* Consider $A \subset X$ such that $\mu(A) > 0$.

   Let $R_n = \{a \in A | f^n(a) \in A\} = A \cap f^{-n}(A)$.

   We have: $a \in A$ returns to $A \Leftrightarrow a \in A$ and $f^n(a) \in A$ for some $n \in \mathbb{N} \Leftrightarrow a \in R_n$ for some $n \in \mathbb{N} \Leftrightarrow a \in \bigcup_{n=1}^{\infty} R_n$.

   Thus, we obtain that the set R of the points of A that return to A coincides with $\bigcup_{n=1}^{\infty} R_n$.

   Assume, for contradiction, that $\mu(R_n) = 0$ for all $n \in \mathbb{N}$. Then, since we are in the setting of the Poincaré Recurrence Theorem, we have

   $$\mu(R) = \mu(A).$$

   Also, we have that

   $$\mu(R) = \mu(\bigcup_{n=1}^{\infty} R_n) \leq \sum_{n=1}^{\infty} \mu(R_n) = 0,$$

   so $\mu(R) = 0$.

   Thus, we obtain that $\mu(A) = 0$ contradicting the fact that $\mu(A) > 0$. Hence, there exists $n \in \mathbb{N}$ such that $\mu(A \cap f^{-n}(A)) > 0$. □

6. *Solution.* Notice that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for all $n \in \mathbb{N}$. Then, we have that $f^n(x, y) = (x + ny, y)$ for all $n \in \mathbb{N}$.

   Consider the set $B = [0, 1] \times [1, 2]$. Then, B is measurable and $\mu(B) = (1 - 0) \cdot (2 - 1) = 1$ where μ is the Lebesgue measure on $\mathbb{R}^2$. We show that for any $(x, y) \in B \setminus \{(0, 1)\}$ there is no

$n \in \mathbb{N}$ such that $f^n(x, y) \in B$. To have $(x, y) \in B$ and $f^n(x, y) \in B$ where $n \in \mathbb{N}$, we have to have $0 \le x \le 1$, $1 \le y \le 2$, and $0 \le x + ny \le 1$. In particular, $-1 \le -\frac{x}{y} \le n \le \frac{1-x}{y} \le 1$. Therefore, the only option for $n \in \mathbb{N}$ is that $n = 1$ which implies that $\frac{1-x}{y} = 1$ so $x + y = 1$ which implies that $x = 0$ and $y = 1$ as $x \in [0, 1]$ and $y \in [1, 2]$. Notice that $(0, 1) \in B$ and $f(0, 1) = (1, 1) \in B$. Therefore, the measure of the set of the points in B that return to B is equal to the measure of the set $\{(0, 1)\}$ which is equal to 0 and isn't equal to the measure of B. $\square$