1. CM

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$\vec{r}$ refers to position ~~WRT 6?~~

$\vec{l}$ refers to angular momentum or position.

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https// sites. google. com/a/ stonybrook. edu/ daverin - phy501.

review of Newtonian mechanism.

$$\begin{cases} \vec{F} = m\ddot{\vec{r}} \\ \vec{F}_{12} = -\vec{F}_{12} \\ \vec{F}_{12} \text{ and } \vec{F}_{21} \text{ central.} \end{cases}$$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{u}$

平移不变性?

example : $\cdot\, m(t)$.
$$\vec{F} = \frac{d}{dt}\left(m(t)\frac{d\vec{l}}{dt}\right). \quad ✓$$
$$= \frac{d^2}{dt^2}(m\vec{l}) \quad \times$$
$$= m\frac{d^2}{dt^2}\vec{l} \quad \times.$$

thus if we let outer force $\vec{F}=0$ then we find a conserved quantity : $m(t)\frac{d\vec{l}}{dt}$ or $\sum m_i \vec{v}_i = \vec{P}$ (thats also an ategrommy conserved)

centre of mass : $\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$

$$\begin{cases} m_i\frac{d^2}{dt^2}\vec{r}_i = \vec{F}_i + \sum_j \vec{f}_{ji} \\ \frac{d^2}{dt^2}(\sum m_i \vec{R}) = \sum \vec{F}_i \end{cases} \Rightarrow \text{momentum conserved while } \vec{F} = \sum\vec{F}_i = 0$$

~~angula~~ angular momentum : $\vec{J} = \sum m_i \vec{r}_i \times \dot{\vec{r}}_i$

$$\begin{cases} d_t\vec{J} = \sum_i \vec{r}_i \times \frac{d}{dt}(m_i\dot{\vec{r}}_i) = \sum \vec{r}_i \times (\vec{F}_i + \sum \vec{f}_{ij}). \\ \sum \vec{r}_i \times \sum \vec{f}_{ij} = \frac{1}{2}\sum(\vec{r}_i - \vec{r}_j)\times \vec{f}_{ij} = \frac{1}{2}\sum |\vec{r}_i - \vec{r}_j|\, \hat{r}_{ij} \times \vec{f}_{ij} = 0. \text{ Central!!} \end{cases}$$

Ch1. Lagrangian mechanism

$\{q_i\} \triangleq q$.
$$\mathcal{L} = \mathcal{L}(\dot{q}, q, t) \quad \sim \text{Lagrangian}$$
$$S = \int dt\, \mathcal{L} \quad \sim \text{~~motion~~ action}$$

☆ ☆ and actual motion refers to S minimized trajectory

remark: the variable of S without using the minimized trajectory is a large part that is the same quantity compared with R, say $q(t)$. the dimension?

$$\delta S = \int dt\left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \quad \text{Lagrangian equation of motion}$$

remark 2: Lagrangian is not unique. (for the unique ~~motion~~ equation).

$$\Leftrightarrow \quad L' = L + \frac{df}{dt}$$

question: why don't we have $f = f(q, \dot{q}, t)$ but $f = f(q, t)$

clue: $df/dt = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial \dot{q}}\ddot{q} + \frac{\partial f}{\partial t}$ (if $\dot{q}$ is in f).

e.g.1. $L = T - V = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r})$.

$$\Rightarrow \frac{d}{dt}(m\dot{x}_i) = -\frac{\partial V}{\partial x_i} = F_i$$

e.g.2. $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$.

$$\Rightarrow \begin{cases} \frac{d}{dt}(m\dot{r}) = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2 \\ \frac{d}{dt}(mr^2\dot{\theta}) = -\frac{\partial V}{\partial \theta} \; (= rF_\theta) \end{cases}$$

e.g.3. $X = X(q_1, \dots q_n, t)$

$$\dot{X} = \frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial t} \qquad (\dot{X})^2 = \left[\frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial t}\right]^2$$

$$T = \frac{m}{2}(\dot{r})^2 = \sum a_{ij}\dot{q}_i\dot{q}_j + \sum b_i \dot{q}_i$$

$$-\nabla U = F_q \vec{e}_q$$

e.g.4. Newton's equation for e.g.2. say translation that

$$m\ddot{x} = F_x \quad \text{to} \quad m\ddot{r} = \cdots$$

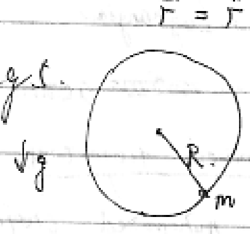
expand $\dot{x}$ and other terms.

(or just calculate $\ddot{\vec{r}}$ to $\vec{e}_r, \vec{e}_\varphi, \vec{e}_\theta$ and $d\vec{e}_r, d\vec{e}_\varphi, d\vec{e}_\theta$?)

$$\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\,\dot{\varphi}\,\vec{e}_\varphi$$

$$\ddot{\vec{r}} = \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + \dot{r}\dot{\theta}\vec{e}_\theta + r\ddot{\theta}\vec{e}_\theta + r\dot{\theta}\dot{\vec{e}}_\theta + (\dot{r}\sin\theta\dot{\varphi} + r\cos\theta\dot{\theta}\dot{\varphi} + r\sin\theta\ddot{\varphi})\vec{e}_\varphi + r\sin\theta\dot{\varphi}\dot{\vec{e}}_\varphi$$

e.g.5.



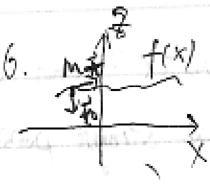
① $L = \frac{m}{2}(R\dot{\theta})^2 + mgR\cos\theta$.

$$\Rightarrow mR^2\ddot{\theta} = -mgR\sin\theta.$$

② full version: $\begin{cases} L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta + \lambda(r - R) = L(r, \dot{r}, \theta, \dot{\theta}, \lambda, t). \\ r - R = 0. \end{cases}$

$$\Rightarrow \begin{cases} \frac{H}{R} = \lambda = \text{const}. \\ mR^2\ddot{\theta} + mgR\sin\theta = 0. \end{cases}$$

w+f is the potential introduced here?

☆ ~~e.g.~~ e.g. 6.  ① $z=f(x)$. flat curve

$d\vec{l} = (1, f')\,dx$. $dl = dx(1+f'^2)^{1/2}$

$l = \int_0^x \sqrt{1+f'^2}\,dx$. ($\underline{l \text{ is a coordinate}}$).

$\mathcal{L} = \frac{m}{2}\dot{l}^2 - \underbrace{U(l)}$!! $\Rightarrow m\ddot{l} = -\partial_l U$

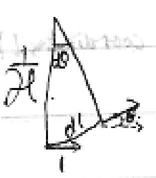
(this form of $\mathcal{L}$ is not so general for quantum mechanics ??.

$\vec{t} \equiv \frac{d\vec{l}}{dl} = \frac{1}{(1+f'^2)^{1/2}}(1, f')$ (inertial time ?)

$\frac{d\vec{t}}{dl} = \left(-\frac{f''f'}{(1+f'^2)^{3/2}}, -\frac{f''f'^2}{(1+f'^2)^{3/2}} + \frac{f''}{(1+f'^2)^{1/2}}\right)\frac{1}{\sqrt{1+f'^2}}$

$= \frac{1}{\sqrt{1+f'^2}}(-f', 1)\underbrace{\frac{f''}{(1+f'^2)^{3/2}}} \rightarrow \mathcal{H}$. (called $\underline{\text{curvature}}$).

$\Rightarrow \frac{d\vec{t}}{dl} = \mathcal{H}\vec{n}$ $|\vec{n}|=1$ $\vec{n}\cdot\vec{t}=0$.



$dl = \vec{\mathcal{H}}\cdot d\theta$.

$d\theta = |\Delta(\vec{t})|$ $\underline{\underline{\vec{F}_c /\!/ \vec{n}}}$.

② $\mathcal{L} = \frac{m}{2}(\dot{x}^2+\dot{z}^2) - U(x,z) + \lambda(z-f(x))$.

$$\Rightarrow \begin{cases} m\ddot{x} = -\partial_x U - \underline{\lambda f'} \\ m\ddot{z} = -\partial_z U + \underline{\lambda} \\ z = f(x) \end{cases} \Rightarrow \begin{cases} \vec{F}_c = (-\lambda f', \lambda) = F_c\vec{n} \\ F_c = \lambda\sqrt{1+f'^2}. \end{cases}$$

$\Updownarrow$ but how? $\Longrightarrow \overset{(1)}{m}(\ddot{x} + f''f'\dot{x}^2 + (f')^2\ddot{x}) = -\partial_x U - f'\partial_z U = -d_x U$.

$$\begin{cases} ~~m\ddot{x}~~ \; m\ddot{l} = -\dot{U} \\ F_c = \lambda\sqrt{1+f'^2} \end{cases}$$

then. $\begin{cases} \dot{l} = (1+f'^2)^{1/2}\dot{x} \\ \ddot{l} = (1+f'^2)^{1/2}\ddot{x} + \frac{f'f''\dot{x}^2}{(1+f'^2)^{1/2}}. \end{cases}$

$\Rightarrow m(1+f'^2)^{1/2}\ddot{l} = -\frac{dU}{dx}$.

question2: what if we expand U to complex field. i.e.

$m(\ddot{x}+i\ddot{z}) = -\partial_x U - i\partial_z U - \lambda f' + i\lambda$. ?, (non sense).

any conclusion ?

(2). $\lambda = m\ddot{z} + \partial_z U = m\frac{f''\dot{x}^2}{1+f'^2} + \underbrace{\frac{1}{1+f'^2}(\partial_z U - f'\partial_x U)}$ $\nabla U\cdot\vec{n}$

$F_c = m\dot{l}^2\mathcal{H} + \nabla_x U\cdot\vec{n} = \underbrace{m\dot{l}^2}_{mv^2}\mathcal{H} - \vec{F}\cdot\vec{n}$.

question: what of the motion has a vertex? (not smooth)
clue: you may find a ~~solution~~ model to that strange point?

Ch2. Conservation Laws.

① Energy. $\frac{\partial L}{\partial t} = 0$. then

$$\frac{d}{dt}L = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{d}{dt}(p\dot{q} - H). \quad p = \frac{\partial L}{\partial \dot{q}}$$

$$\frac{d}{dt}H = \frac{d}{dt}(p\dot{q} - L) = 0.$$

think of $L' = L + d_t f(q) + \text{const}$. $\quad d_t f = \partial_q f \cdot \dot{q}$

$$H' = \dot{q}\frac{\partial L'}{\partial \dot{q}} - L' = \dot{q}\frac{\partial L}{\partial \dot{q}} - L - \text{const} = H - \text{const}.$$

(this operation just gives out the relation once we transform the side of a defination of L itself rather than some coordinates!)

for $L = T - V$. $\quad T = \frac{1}{2}\sum a_{ij}(q)\, \dot{q}_i \dot{q}_j$

$$H = 2T - T + V = T + V.$$

② momentum. transformation $\vec{r} \to \vec{r} + \vec{\varepsilon}$

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial t}\delta t = \frac{\partial L}{\partial \vec{r}}\vec{\varepsilon} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}}\cdot\vec{\varepsilon}\right) = 0.$$

$$\Rightarrow \vec{p} = \sum\frac{\partial L}{\partial \dot{\vec{r}}} = \text{const} = \sum m_i \dot{\vec{r}}_i = \frac{d}{dt}(\mu\vec{R}) = \frac{d}{dt}(m_i \vec{r}_i) = \mu\vec{V} \quad \mu = \sum m_i$$

$p_j = \frac{\partial L}{\partial \dot{q}_j}$ is called generalized momentum.

here we must ask $\sum \vec{F}_i = -\sum \frac{\partial U}{\partial \vec{r}_i} = \sum \frac{\partial L}{\partial \vec{r}_i} = 0$.

$$E = \frac{1}{2}m_i(\vec{v}_i' + \vec{V})^2 + U = E' + \frac{1}{2}\mu\vec{V}^2$$

(we rewrite it within the c.m. coordinate.

③ angular momentum. transformation $\vec{r} \to \vec{\theta} \times \vec{r}$

③ angular momentum. $\delta\vec{x} = \delta\vec{\varphi} \times \vec{x} \quad \delta\vec{v} = \delta\vec{\varphi} \times \vec{v}$

$$\delta L = \frac{\partial L}{\partial \vec{r}} \cdot \delta\vec{r} + \frac{\partial L}{\partial \vec{v}} \delta\vec{v} = \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) \cdot \delta\vec{r} + \left(\frac{\partial L}{\partial \vec{v}}\right) \cdot \frac{d}{dt}(\delta\vec{r})$$

$$= \frac{d}{dt}(\vec{p} \cdot \delta\vec{r}) = \frac{d}{dt}(\delta\vec{\varphi} \cdot (\vec{r} \times \vec{p})) \triangleq \frac{d}{dt}(\delta\vec{\varphi} \cdot \vec{J}).$$

the structure $\int \vec{p} \cdot d\vec{x}$ exist!

for $\vec{r} = \vec{r}' + \vec{a}$. then. $\vec{J} = \vec{J}' + \vec{a} \times \vec{p}$

for $\vec{a} = \vec{a}(t)$ then $\vec{J} = \vec{r} \times \vec{p} = (\vec{r}' + \vec{a}) \times (\vec{p}' + \mu\vec{V})$.

$$= \vec{J}' + \vec{a} \times \vec{p}' + \vec{a} \times \mu\vec{V} + \vec{r}' \times \mu\vec{V}$$

for c.m.s. $\vec{p}' = 0 \Rightarrow \vec{J} = \vec{J}' + \vec{R} \times \mu\vec{V} = \vec{J}' + \vec{R} \times \vec{p}$

therm. $\frac{\partial L}{\partial \dot{\varphi}} = J_z = m r^2 \dot{\varphi}$

QISM

quantum inverse scattering

④ 1D. problem.

$L = \frac{1}{2} a(q) \dot{q}^2 - U(q)$. we could ask $dx = \sqrt{\frac{a(q)}{m}}\, dq$.

$\Rightarrow L = \frac{1}{2} m v^2 - U(x). \Rightarrow m\ddot{x} = -\frac{dU}{dx}. \Rightarrow m \frac{1}{2} \frac{dv^2}{dt} = -\frac{dU}{dt}.$

$\Rightarrow E = \frac{m}{2} v^2 + U(x) = \text{const} \Rightarrow \dot{x} = \sqrt{\frac{2}{m}(E - U)} \Rightarrow dt = \sqrt{\frac{m}{2}} \cdot \frac{dx}{\sqrt{E-U}}$

period within a T. is:

$$T = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E-U}} \triangleq T(E) \text{ as } E \text{ varies}$$

$$T(E) = \sqrt{2m}\left[\int_0^{x_2} \frac{dx}{\sqrt{E-U}} - \int_0^{x_1} \frac{dx}{\sqrt{E-U}}\right].$$

$$= \sqrt{2m} \int_0^E \frac{dU}{\sqrt{E-U}} \left[\frac{dx_2}{dU} - \frac{dx_1}{dU}\right].$$

a trick. $\int_0^{\varepsilon} \frac{dE\, T(E)}{\sqrt{\varepsilon - E}} = \sqrt{2m} \int_0^{\varepsilon} dE \int_0^E \frac{dU}{\sqrt{(\varepsilon-E)(E-U)}} \left[\frac{dx_2}{dU} - \frac{dx_1}{dU}\right]$

$$= \sqrt{2m} \int_0^{\varepsilon} dU \left(\frac{dx_2}{dU} - \frac{dx_1}{dU}\right) \int_U^{\varepsilon} \frac{dE}{\sqrt{(\varepsilon-E)(E-U)}}$$

$$\Rightarrow x_2(\varepsilon) - x_1(\varepsilon) = \frac{1}{\sqrt{2m}\,\pi} \int_0^{\varepsilon} \frac{dE\, T(E)}{(\varepsilon - E)^{1/2}}$$

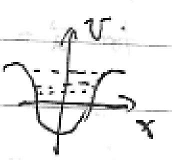
remark: in this derivation, the x_2 or x_1 has been replaced by U and to use $E = f \cdot U(x_i)$. the integration has been taken to $\int_0^E$. then. it's nothing with the particular value of x_1 or x_2.

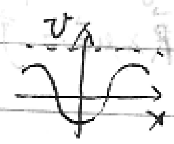
question4: the trick presented here could be treated as some kind of Fourier transformation? 广义变积分变换? General Integration Transformation?

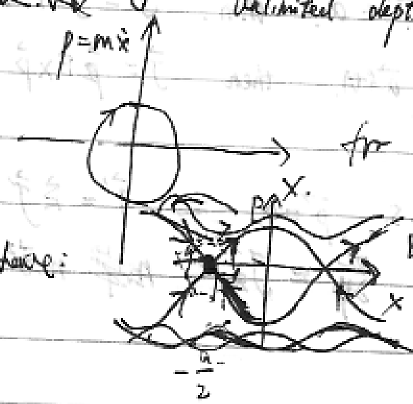
e.g. 1. $U = a|x|$. $\Rightarrow \begin{cases} x_2 = \frac{1}{\sqrt{2m}\cdot 2\pi}\int_0^U dE \frac{T(E)}{\sqrt{U-E}} = aU. \\ T(E) \propto a\sqrt{E} \end{cases}$

e.g. 2. $T(E) \propto \frac{1}{E}$. $x \propto \cancel{\sqrt{E}} \cdot U^0$. unlimited depth.

e.g. 3. $U = -U_0 \cos\frac{2\pi x}{a}$.

for 

and for 

here: 

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{\sqrt{E-U}} \cdot \sqrt{\frac{m}{2}}$$

$$\underset{=}{\overset{E=U_0}{}} \int_{x_0}^{x} \sqrt{\frac{m}{4U_0}} \cdot \frac{dx}{\cos\frac{\pi x}{a}} = \frac{a}{2\pi}\sqrt{\frac{m}{U_0}} \cdot \frac{1}{2} \ln \frac{1+\sin\frac{\pi x}{a}}{1-\sin\frac{\pi x}{a}} \Bigg|_{x_0}^{x}$$

$$\underline{\underline{x_0 = 0 \quad x \to \frac{a}{2}}} \; \frac{a}{4\pi}\sqrt{\frac{m}{U_0}} \ln \frac{2}{1-\sin\frac{\pi x}{a}} \to +\infty !!$$

e.g. 4. $U = -kx^2/2$

⑤ Central force problem.

$$L = \frac{m_1}{2}\dot{\vec{r}}_1^2 + \frac{m_2}{2}\dot{\vec{r}}_2^2 - U(|\vec{r}_1 - \vec{r}_2|)$$

$$\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 / m_1 + m_2. \quad \vec{r} = \vec{r}_1 - \vec{r}_2 \Rightarrow \begin{cases} \vec{r}_1 = \vec{R} + \frac{m_2}{m_1+m_2}\vec{r} \\ \vec{r}_2 = \vec{R} - \frac{m_1}{m_1+m_2}\vec{r} \end{cases}$$

$$\Rightarrow L = \frac{m_1+m_2}{2}\dot{\vec{R}}^2 + \frac{m_1 m_2}{2(m_1+m_2)}\dot{\vec{r}}^2 - U(|\vec{r}|).$$

~~coord~~ different parts of motion are ~~is~~ ~~de~~ separated in Lagrangian with the "add" thing say $L = L_1 + L_2$

(b) go with the part (b). in former page.

$$\vec{L} = \sum m_i \vec{r}_i \times \dot{\vec{r}}_i = (m_1+m_2)\vec{R}\times\dot{\vec{R}} + \mu \vec{r}\times\dot{\vec{r}} = m\vec{R}\times\dot{\vec{R}} + \underline{\mu\vec{r}\times\dot{\vec{r}}}$$

since we have "fix" the central point of the system, then the position transformation which could derive momentum would not work as before. In other words, we have introduced sth. like extra force to the system. and momentum would not conserve at all. ?

if $U(\vec{r}) = U(r)$. then:

$$\begin{cases} m\ddot{r} = mr\dot\varphi^2 - U' \\ mr^2\dot\varphi = \text{const} = l \end{cases} \quad \left(E = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{1}{2}mr^2\dot\varphi^2}_{\uparrow} + U = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + U\right)$$

trick: pay attention to the signal !!

$$\Rightarrow t - t_0 = \int_{r_0}^{r} \frac{dr}{\sqrt{E - U - \frac{l^2}{2mr^2}}} \quad \text{and} \quad \dot\varphi = l/mr^2$$

question: Why changing the frame to (r) instead of (r, θ) gives a potential that is $\propto r^{-2}$ form? Is there anything that's ~~deep~~ profound in the form?

$$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{dr/r^2}{\sqrt{E - U - l^2/2\mu r^2}} \cdot \frac{\sqrt{l^2}}{\sqrt{2\mu}} \xRightarrow{u \triangleq 1/r} d\varphi = -\frac{\sqrt{l^2}\,du}{\sqrt{2\mu}\cdot\sqrt{E - U - l^2u^2/2\mu}}$$

$$\Leftrightarrow (u')^2 = \left(E - U - l^2\frac{u^2}{2\mu}\right)\frac{2\mu}{l^2} \Longrightarrow 2u''u' = \left(-\frac{dU}{du}u' - l^2\cdot\frac{u'}{\mu}u\right)2\mu/l^2$$

$$\Leftrightarrow \cancel{2u'' = -\frac{dU}{du} - \frac{l^2}{\mu}u}. \quad u'' + u = -\frac{\mu}{l^2}\frac{dU}{du} \;!!$$

e.g. (Kepler's problem).

$$U = -\varkappa/r = -\varkappa u \Rightarrow u'' + u = \frac{\mu\varkappa}{l^2}$$

$$\Rightarrow u = \frac{\mu\varkappa}{l^2} + A\cos\varphi = \frac{\mu\varkappa}{l^2}(1 + \varepsilon\cos\varphi) \triangleq \frac{1+\varepsilon\cos\varphi}{p}$$

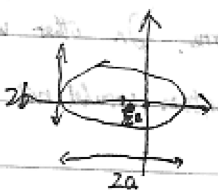
remark: notice that we have $r(\varphi) = r(\varphi + 2\pi)$,

which is not a trivial outcome for random potential!

(Bertrand's theorem: $U \propto 1/r$ or $U \propto r^2$).

the relation of (ε and 1.) is defined by (E and 0.) say.

$$\varepsilon^2 = 1 + 2\frac{\ell^2 E}{\mu \mathcal{H}^2} \begin{cases} > 1. & E > 0 \\ = 1. & E = 0. \\ < 1. & E < 0. \text{ elliptics} \\ = 0 & \text{circle} \end{cases}$$



$$a = \mathcal{H}/2|E|. = p/(1-\varepsilon^2).$$

$$b = \ell/(2\mu|E|)^{1/2}$$

time dependence of the trajectory.

$$t - t_0 \mp \frac{\mu}{2}\int_{r_0}^{r} dr / (E + \mathcal{H}/r - \ell^2/2\mu r^2)^{1/2}$$

$$\Rightarrow \quad = \frac{\mu_0}{2}\int_{r_0}^{r} r\,dr / (E r^2 + \mathcal{H} r - \ell^2/2\mu)^{1/2}$$

now let $r = a(1 - \varepsilon\cos\xi)$.

remark: there is sth. different for ξ and φ. although we have:

$$\begin{cases} \xi = 0 \\ \xi = \pi \end{cases} \Leftrightarrow \begin{cases} \varphi = 0 \\ \varphi = \pi. \end{cases}$$ they act in different ways.

have:

$$\frac{\ell^2}{2\mu} = \frac{\mathcal{H}}{2}a(1-\varepsilon^2) \qquad E r^2 = -\frac{a\mathcal{H}}{2}(1-\varepsilon\cos\xi)^2 \qquad \mathcal{H} r = \mathcal{H} a(1-\varepsilon\cos\xi),$$

$$\Rightarrow \Delta t = \sqrt{\frac{\mu}{2}}\int_{r_0}^{r} a(1-\varepsilon\cos\xi)\, a\varepsilon\sin\xi\, d\xi \Big/ \sqrt{\frac{\mathcal{H}}{2}}\, a\varepsilon\sin\xi$$

$$= \sqrt{\frac{\mu}{\mathcal{H}}}a^3(\xi - \varepsilon\sin\xi).$$

$$\Rightarrow T = 2\pi\left(\frac{\mu a^3}{\mathcal{H}}\right)^{1/2}$$

⑥. Runge - Lenz vector

let $\vec{p} = \mu\dot{\vec{r}}$ $\quad (\dot{\vec{p}} = -U'\hat{r})$.

$$\Rightarrow \dot{\vec{p}} \times \vec{M} = \frac{d}{dt}(\vec{p}\times\vec{M}) = -U'\hat{r}\times(\vec{r}\times\mu\dot{\vec{r}})$$

$$= \mu U' r^2 \dot{\hat{r}} = \mu U' r^2 \vec{e}_\theta\,\dot{\theta}$$

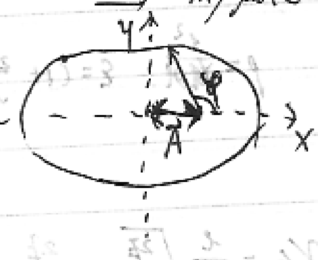
$$\xlongequal{\text{Kepler}} \mu\mathcal{H}\dot{\theta}\,\vec{e}_\theta$$

$$\Rightarrow \vec{p}\times\vec{M} - \mu\mathcal{H}\hat{r} = \text{const} \triangleq \vec{A}$$

now find the meaning of $\vec{A}$.

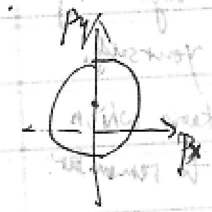
$$\vec{A}\cdot\vec{A} = (\vec{p}\times\vec{M})^2 + \mu^2\alpha^2 \cancel{=p^2M^2+\mu^2\alpha^2} - 2(\vec{p}\times\vec{M})\cdot\mu\alpha\hat{r}$$
$$= p^2M^2 + \mu^2\alpha^2 - 2\mu\alpha\frac{M^2}{r}$$
$$= \mu^2\alpha^2\cdot\Big[1+\frac{2M^2}{\mu\alpha^2}\underbrace{\Big(\frac{p^2}{2\mu}-\frac{\alpha}{r}\Big)}_{E}\Big] = \mu^2\alpha^2\varepsilon^2 \quad \text{(amazing)}.$$

$$\vec{A}\cdot\vec{r} = -\mu\alpha r + M^2 \overset{\Delta}{=} \varepsilon\mu\alpha r\cos\varphi.$$
$$\Longrightarrow M^2/\mu\alpha r - 1 = \varepsilon\cos\varphi. \Rightarrow r = \frac{M^2}{\mu\alpha}\cdot\frac{1}{1+\varepsilon\cos\varphi} \quad !!$$



besides, we $p_x^2 + (p_y - \frac{A}{M})^2 = (\frac{\mu\alpha}{M})^2$

amazing!

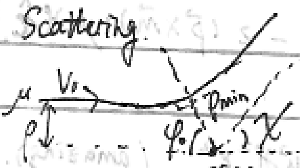


?

Question 6.: Is there any relation between the $(\vec{p}-\vec{A}/c)$ and the electromagnetic ~~for~~ charged particle's Lagrangian?

O.
need to fill.
10.7.

Scattering.



$$\frac{d\sigma}{d\chi} = \frac{\rho}{\sin\chi}\left|\frac{d\rho}{d\chi}\right|. \longrightarrow \text{know } \rho(\chi).$$

☆. do this again by yourself! know which to remember.

In Rutherfold model, we should have

$$V(r) = \alpha/r. \quad (\alpha > 0).$$

and in this particular situation,

$$r = p/(\varepsilon\cos\varphi - 1). \qquad p = \frac{l^2}{\cancel{\mu}\alpha}. \quad \varepsilon = \left(1 + \frac{2l^2E}{\mu\alpha^2}\right)^{1/2}$$

$$\cos\varphi_0 = 1/\varepsilon$$

$$\chi = \pi - 2\varphi_0.$$

$$\Longrightarrow \sin\chi/2 = \cos\varphi_0 = 1/\varepsilon. \quad \text{and} \quad \cot\chi/2 = \frac{l}{\alpha}\sqrt{\frac{2E}{\mu}} = \frac{2E}{\alpha}\rho$$

$$\Longrightarrow \frac{d\sigma}{d\chi} = \left(\frac{\alpha}{2E}\right)^2 \cot\chi/2 \,/\, \sin\chi \cdot \left|-\frac{1}{\sin^2\chi/2}\right| \cdot \frac{1}{2}$$

$$= \left(\frac{\alpha}{4E}\right)^2 \cdot \left(\sin\frac{\chi}{2}\right)^{-4}$$

the most important practical explaination is at $\chi = \pi$. not vanishing!

another e.g: rigid ball. (hard sphere).

$$V = \begin{cases} \infty & r < a \\ 0 & r = a. \end{cases}$$

$$\chi = \pi - 2\varphi_0. \qquad \rho = a\sin\varphi_0 = a\cos\chi/2$$

$$\Longrightarrow \frac{d\sigma}{d\chi} = a^2/4. \quad \text{unifornly}$$

the meaning of total cross-section is:

$$\sigma = \int \frac{d\sigma}{d\chi}\cdot d\Omega = \int 2\pi\rho\, d\rho$$

namely "effective scatter area".

for general cases: $\dot\varphi = \frac{l}{\mu r^2}$ $\quad d\varphi = \frac{dr}{v_r}\cdot\frac{l}{\mu r^2}$

$$\Longrightarrow \varphi_0 = \int_{r_{min}}^{+\infty} \frac{\rho\, dr/r^2}{\sqrt{1 - \rho^2/r^2 - V/E}} \qquad \left(\text{here } \rho^2 = \frac{l^2}{2\mu E}\right)$$

next is to do some perturbation.

external force for Larmour

for general cases, calculate the term in perturbation series.

case ①. $V=0$. $r_{min}=\rho$.

$$\varphi_0=\frac{\pi}{2}.$$

case ②. 1st order.

$$\varphi_0 \approx \int \frac{\rho}{r^2}dr\left[\frac{1}{\sqrt{1-\rho^2/r^2}}+\frac{V/E}{2(1-\rho^2/r^2)^{3/2}}\right]$$

$$\chi=-\frac{\rho}{E}\int_\rho^{+\infty}\frac{V\,dr/r^2}{(1-\rho^2/r^2)^{3/2}}=+\frac{\rho}{E}\int_\rho^{+\infty}V\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2-\rho^2}}\right)dr$$

$$=-\frac{\rho}{E}\int_\rho^{+\infty}\frac{dr}{\sqrt{r^2-\rho^2}}\frac{\partial V}{\partial r}+\infty.$$

☆ an amazing method: $\begin{cases}P_y=0\\ P_x=mV_0\end{cases}$ $P_y=\int dt\,F_y=-\frac{\rho}{V_0}\int\frac{dr}{\sqrt{r^2-\rho}}\cdot\frac{dV}{d\varphi}$

$$\Rightarrow \chi\approx\frac{P_y}{P_x}=-\frac{\rho}{E_0}\int_\rho^{+\infty}\frac{dr}{\sqrt{r^2-\rho^2}}\cdot\frac{dV}{dr}\quad(\text{small angle.})$$

if $V\sim r^{-n}$ then $\chi\sim\rho^{-n}$

$$\sigma(\chi)\simeq\frac{\rho}{\chi}\left|\frac{d\rho}{d\chi}\right|=\chi^{-2-2/n}$$

Ch5. Small oscillations

for 1-D condition, we have:

$$L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}Ux^2$$

$$\Rightarrow m\ddot{x}=-Ux$$

$$\Rightarrow x=A\cos\omega t+B\sin\omega t.\quad\left(\omega^2=\frac{u}{m}\right).$$

$$=x_0\cos\omega t+(V/\omega)\sin\omega t$$

$$=\mathrm{Re}\left((x_0-i\tfrac{V_0}{\omega})e^{i\omega t}\right)=a\cos(\omega t+\varphi).$$

$$\begin{cases}a^2=x_0^2+\frac{V_0^2}{\omega^2}\\ \tan\varphi=-\frac{V_0}{\omega x_0}\end{cases}$$

$$\text{and } E=\frac{1}{2}mv^2+\frac{1}{2}kx^2=m\omega^2a^2/2$$

now think of external force:

$$L=T-U_{os}+F(t)\cdot x$$

$$\Rightarrow m\ddot{x}+ux=F(t)$$

☆ now introduce the following external force.

$$\xi(t)=\dot{x}+i\omega x$$

$\Rightarrow (\partial_t + i\omega)\xi = F(t)/m$ (first order df).

$\Rightarrow \xi = \xi_0 e^{i\omega t} + \int_{-\infty}^{t} \frac{F(t')}{m} e^{i\omega(t-t')} dt'$

think of $t \to -\infty$. $\xi = \xi_0 e^{i\omega t} = 0. \Rightarrow \xi_0 = 0$.

and try to look at $F(t)$.

$\xi \approx \frac{e^{i\omega t}}{m} \int_{-\infty}^{+\infty} F(t') e^{-i\omega t'} dt'$.

in another way,

$E \triangleq \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega^2 x^2$

$= \frac{m}{2} |\xi|^2$

$\left(= \frac{1}{2m} \left| \int_{-\infty}^{+\infty} F(t) e^{i\omega t} dt \right|^2 \right)$

e.g. $F(t) = F_0 / \cosh(t/\tau)$.

calculate $F \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\cosh(t/\tau)} dt$

$\xlongequal[t/\tau = x]{\omega\tau = \lambda} \tau \int_{-\infty}^{+\infty} \frac{e^{i\lambda x}}{\cosh x} dx$.

(notice here, $\oint \frac{f(z)}{g(z)} dz = 2\pi i \frac{f(z_i)}{g'(z_i)}$ not $f(z_i) \cdot 2\pi i$)

now use the residue theorem, we get:

$ans = \tau \cdot 2\pi i \cdot \frac{1}{i} \sum (-1)^j e^{i\lambda x_j} \Big|_{x = i\frac{\pi}{2}, i\frac{3}{2}\pi, \ldots\ldots}$

$= \tau \cdot 2\pi\, e^{-\frac{\pi}{2}\lambda} \cdot \frac{1}{1+e^{-\pi\lambda}} = \tau \cdot \pi \cdot \frac{1}{ch(\pi\lambda/2)}$

$E = \frac{F_0^2}{2m} \cdot \frac{\pi^2\tau^2}{\cosh^2(\pi\omega\tau/2)}$

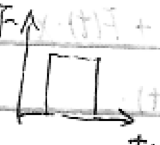
if the pulse is short enough, say $\omega\tau \ll 1$. then.

$E = (F_0^2/2m) \cdot \pi^2\tau^2 \cdot \left(\approx \frac{p^2}{2m} \right)$.

if $\tau\omega \gg 1$. then it's an adiabatic process.

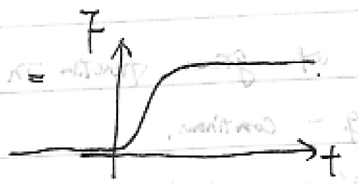
$E = (F_0^2/m)\, 2\pi^2\tau^2 \cancel{e^{-\pi\omega\tau}} e^{-\pi\omega\tau}$

e.g. 2. (precise) $F(t) = F_0$



13.

damping and green function

e.g. 3. $F(t) =$ 

$$E = \cancel{\tfrac{1}{2}k^2x^2} + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2$$

$$U = E - F(t)\,x$$

$$\Rightarrow dE = F\,dx \quad \Rightarrow dU = -X\,dF$$

$$\dot{U} = -\dot{x}F = -\dot{F}\frac{1}{\omega}\,\mathrm{Im}\,\xi$$

? $\vec{\mu}\cdot\vec{B}$ $B(t)$ adiabatic

SGM δ. information

and $X(t) = \frac{1}{\omega}\mathrm{Im}\,\xi = \frac{1}{\omega m}\mathrm{Im}\left[\frac{1}{-i\omega}\left(F(t) - \int_{-\infty}^{t} dt'\, e^{i\omega(t-t')}\dot{F}(t')\right)\right]$

$$\Rightarrow \dot{U} = \frac{-1}{m\omega^2}\left[\frac{1}{2}(\dot{F}^2(t)) - \mathrm{Re}\int_{-\infty}^{t} dt'\,\dot{F}(t)\int_{-\infty}^{t'} dt''\,\dot{F}(t'')\,e^{i\omega(t'-t'')}\right]$$

$$= -\frac{F^2}{2m\omega^2} + \frac{1}{2m\omega^2}\int_{-\infty}^{t}\int_{-\infty}^{t} \underbrace{\dot{F}(t')\dot{F}(t'')\,e^{i\omega(t'-t'')}}\,dt'dt''$$

$$\left|\dot{F}(t)\,e^{i\omega t}\,dt\right|$$

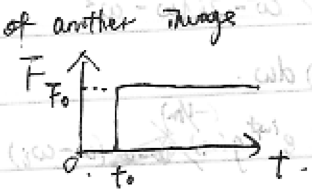
now let $m\omega^2 = k$. $\Rightarrow$ and

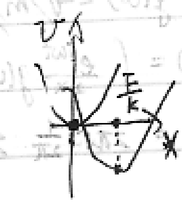
$$X_0(t) = \frac{F(t)}{k}.$$

$$\Rightarrow U(t) + \frac{F^2(t)}{2m\omega^2} = \frac{m\dot{x}^2}{2} + \frac{1}{2}k(x - x_0)^2\Big|_{t\to+\infty} = \frac{1}{2m\omega^2}\left|\int_{-\infty}^{+\infty}\dot{F}(t)\,e^{i\omega t}\,dt\right|^2$$

this is in agreement with what we obtained before !!

think of another image





e.g. $\dot{F} \propto (1 + \tanh(t/\tau))$

$\omega\cdot\tau >> 1.$

a sudden change !!

Now talk about sth. about ~~dissipassion~~ "dissipation".

Ohmic friction:

$$m\ddot{x} + kx = -\eta\dot{x} \qquad 2\lambda = \eta/m.$$

now turn to a out force:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = F/m$$

try to introduce a trick called "Green's function":

$$\ddot{g} + 2\lambda\dot{g} + \omega_0^2 x = \delta(x,t)/m$$

Some properties of green function in this problem

① g – continous.

g' may not be continuous (yet it may be step function)

② retarded term. $g = 0$ for $t \leq 0$.

③ $X(t) = F(t) * g(t) = \int_{-\infty}^{t} F(t') g(t-t') dt'$ $\left(= \int_{-\infty}^{+\infty} F(t') g(t-t') dt' \right)$

remark: δ function is ~~ε~~ in deed a good intepretation of Fourier ~~function~~ transformation for ~~δ~~ $\delta = \int \frac{1}{2\pi} e^{i\omega(t-t')} d\omega$.

~~here we have condition for~~

now turn our eyes to frequency domain. (Fourier series).

$$\begin{cases} g(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} g(\omega). \\ \delta(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \end{cases}$$

Say. $g(t) \to g(\omega)$. $\qquad \dot{g} \to i\omega g(\omega)$

$\delta(t) \to 1$. $\qquad \ddot{g} \to -\omega^2 g(\omega)$.

$$\Rightarrow (-\omega^2 + 2i\lambda\omega + \omega_0^2)\, g(\omega) = \frac{1}{m}.$$

$$\Rightarrow g(\omega) = -(1/m) \Big/ \omega^2 - 2i\lambda\omega - \omega_0^2$$

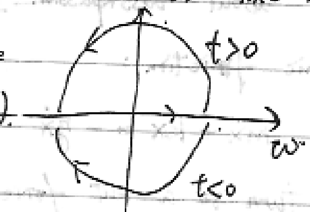
and $g(t) = \int \frac{e^{i\omega t}}{2\pi} g(\omega) d\omega$.

$$= 2\pi i \cdot \frac{1}{2\pi} \Sigma\, e^{i\omega_i t} \cdot \cancel{g(\omega_i)}^{(-1/m)} \Big/ (\omega - \omega_i) \Big|_{\omega_i = \frac{1}{2}(2i\lambda \pm \sqrt{4\omega_0^2 - 4\lambda^2})} .$$

$$\stackrel{\Delta}{=} i\lambda \pm \bar{\omega}$$

tip: the ~~count~~ contour here is like this:

(yet we choose $g(t) = 0$ for $t < 0$).



we should judge which one to complete, since they vary with sign. thus it is a big difference

and seperate the final form with respect to $\omega_0 > \lambda$ or $\omega_0 < \lambda$.

$$\Rightarrow g(t) = i \cdot \left[e^{i(i\lambda + \bar{\omega})t} / 2\bar{\omega} - e^{i(i\lambda - \bar{\omega})t} / 2\bar{\omega} \right] \cdot (-1/m)$$

$$= e^{-\lambda t} \cdot \frac{-i}{m} \frac{1}{2\bar{\omega}} \cdot \left[e^{i\bar{\omega}t} - e^{-i\bar{\omega}t} \right] = + \frac{e^{-\lambda t}}{m} \cdot \frac{1}{\bar{\omega}} \sin\bar{\omega}t.$$

$$\left(= + \frac{e^{-\lambda t}}{m} \frac{\sinh(\bar{\omega} t)}{\bar{\omega}} \right)$$

many body freedom

examples. 1. resonance.

$$F(t) = F_0 \cos\nu t.$$

$$X(t) = \int_{-\infty}^{t} F_0 \cos\nu t \, \frac{\sin\bar\omega(t-\tau)}{m\bar\omega} e^{-\lambda(t-\tau)}$$

$$= \frac{F_0}{m\bar\omega} \cancel{\int} \mathrm{Re} \cancel{\int_{-\infty}^{t}} \int_0^{+\infty} e^{i\nu(t-u)} e^{-\lambda u} \sin\bar\omega u \, du.$$

$$= \frac{F_0}{m\bar\omega} \frac{1}{2} \mathrm{Re} \int_0^{+\infty} (\pm i e^{i\nu(t-u)-\lambda u \mp i\bar\omega u}) \, du.$$

$$= \frac{F_0}{2m\bar\omega} \mathrm{Re}\, e^{i\nu t} \int_0^{+\infty} \pm e^{(-i\nu-\lambda\mp i\bar\omega)u} du.$$

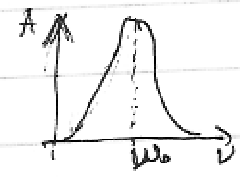
$$= \frac{F_0}{2m\bar\omega} \mathrm{Re}\, e^{i\nu t}/i \cdot \left[\frac{1}{\lambda+i\nu-i\bar\omega} - \frac{1}{\lambda+i\nu+i\bar\omega} \right]$$

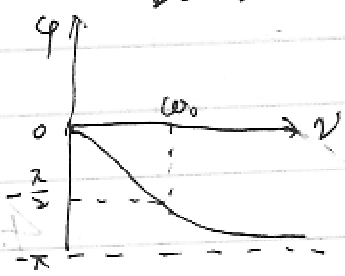
$$= \frac{F_0}{2m\bar\omega} \mathrm{Re} \left(e^{i\nu t} \frac{-2\bar\omega}{\nu^2-2i\lambda\nu-\omega_0^2} \right)$$

$$= \frac{F_0}{m} \frac{\cos\nu t(\omega_0^2-\nu^2) \pm \sin\nu t(2\lambda\nu)}{(\omega_0^2-\nu^2)^2+(2\lambda\nu)^2}.$$

$$\triangleq A \cdot \cos(\nu t+\varphi).$$

$$\begin{cases} A = (F_0/m) \cdot [(\omega_0^2-\nu^2)^2 + 4\lambda^2\nu^2]^{-1/2} \\ \tan\varphi = (\omega_0^2 \cancel{-\nu^2)/2\lambda\nu} - 2\lambda\nu/(\omega_0^2-\nu^2) \end{cases}$$





description: main frequency is still ν.

many degrees of freedom.

1. $U(q_i)$ where ~~equili~~ equilibrum $q_i^{(0)}$

and $x_i = q_i - q_i^{(0)}$

$$\Rightarrow U(q) = U(q_i^{(0)}) + x_i U'(q_i^{(0)}) + \cancel{\tfrac{1}{2} x_i^2 U''(q_i^{(0)})}. \ \tfrac{1}{2}\sum x_i x_j U_{ij}. \quad (U_{ij} = \partial_i\partial_j U|_{q_i^{(0)}})$$

$$\triangleq 0 + \tfrac{1}{2}\sum K_{ij} x_i x_j.$$

property of matrix $\{K_{ij}\}$.

① symmetric $K_{ij} = K_{ji}$.

②. positive. $\sum K_{ij} x_i x_j \geq 0. \quad \forall \{x_i\}.$

2. $T = \frac{1}{2}\sum a_{ij}(\{q\})\cdot \dot{q}_i\dot{q}_j$

remark: adiabatic could be understood by using $\cancel{x}_i(t) = q_i(t) - q_i^{(0)}(t)$.

where $V(q)$ changes with time, yet the change for $q_i^{(0)}$ is so slow that we could just leave it back!

$$T = \frac{1}{2}\sum m_{ij}\dot{x}_i\dot{x}_j$$

property of $[M]$

① $m_{ij} = m_{ji}$

② $\sum m_{ij}\dot{x}_i\dot{x}_j \geq 0$.

$\Rightarrow$ 3. $\Rightarrow \mathcal{L} = T - V$.

$$= \frac{1}{2}\dot{X}^T M \dot{X} - \frac{1}{2} X^T K X$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_k} = \frac{1}{2}\sum m_{ij}\dot{x}_j\delta_{ik} + \frac{1}{2}\sum m_{ij}\dot{x}_i\delta_{jk}$$

$$= \sum \frac{1}{2} m_{kj}\dot{x}_j + \frac{1}{2}\sum m_{ik}\dot{x}_i$$

$$= m_{kj}\dot{x}_j$$

Same way $\frac{\partial \mathcal{L}}{\partial x_k} = -k_{kj}x_j$

$\Rightarrow \quad m_{ij}\ddot{x}_j + k_{ij}x_j = 0$.

4. now let $x_j = a_j\exp(i\omega t)$. (don't know if $a_j \in R$).

$\Rightarrow (-\omega^2 m_{ij} + k_{ij})a_j = 0$.

(*) then. $\| -\omega^2 m_{ij} + k_{ij}\| = 0$.

$\Leftrightarrow \{a_j\}$ has non-0 solution.

(*) is a polynomial (n-order) for ω^2

5. Confirm $\vec{a} \in R^n$

for $\vec{a}^{+}(K - \omega^2 M)\vec{a} = 0$

$= \vec{a}^{+}K\vec{a} - \omega^2\,\vec{a}^{+}M\vec{a}$ $(\in R) \Rightarrow \vec{a}$ could be real, no contradictions!

by the way, $\omega^2 = \frac{\vec{a}^{+}K\vec{a}}{\vec{a}^{+}M\vec{a}}$ (>0).

for each $\lambda_\alpha = \omega_\alpha^2$ we have $\vec{a}_\alpha$ to be unified.

then let $A = \cancel{(a_1 \cdots a_n)}$ $(\vec{a}_1 \cdots \vec{a}_n) \Rightarrow$

17.

$$KA = \cancel{A} MA\lambda$$

$$A^T K = \lambda A^T M \cancel{A}$$

here unification needs to modify.

☆ let $A^T M A = 1$ (another is $(K-\lambda M)A \overset{?}{=} 0$).

$\Rightarrow A^T K A = A^T(MA\lambda) = \lambda$. !!

now let $X = A\cdot Q$. $Q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$. $q_i = q_{i0} e^{i\omega_i t}$

$$\Rightarrow L = \frac{1}{2}(\dot{X}^T M \dot{X} - X^T K X)$$

$$= \frac{1}{2}(\cancel{\dot{Q}^T K M Q}\ \dot{Q}^T A^T M A \dot{Q} - Q^T A^T M A Q).$$

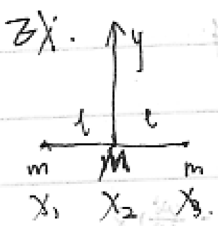
$$= \frac{1}{2}(\dot{Q}^T \dot{Q} - Q^T \lambda Q). \quad !!$$

this is totally orthogonal !!

~~Supply~~

Supplement 10.31.

例.



$\sum m_j \vec{u}_j = 0$. $\vec{r}_j = \vec{r}_j^{(0)} + \vec{u}_j$

$$\vec{M} = \sum m_j \vec{r}_j \times \dot{\vec{r}}_j = \text{const}$$

$$\approx \sum m_j \vec{r}_j^{(0)} \times \dot{\vec{u}}$$

$m(x_1+x_3) + M x_2 = 0$. $\quad (x_2 = -\lambda(x_1+x_3))$.

?

$$\begin{cases} V = \frac{k}{2}[(x_1-x_2)^2 + (x_2-x_3)^2] \\ T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_3^2) + \frac{M}{2}\dot{x}_2^2 \end{cases}$$

$$\begin{cases} q_1 = \frac{1}{\sqrt{2}}(x_1+x_3) \\ q_2 = \frac{1}{\sqrt{2}}(x_1-x_3) \end{cases} \qquad \begin{cases} x_1 = \frac{1}{\sqrt{2}}(q_1+q_2) \\ x_3 = \frac{1}{\sqrt{2}}(q_1-q_2) \end{cases}$$

$$\Rightarrow V = \frac{k}{2}[q_2^2 + (1+2\lambda)^2 q_1^2]$$

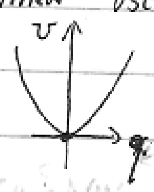
$$T = \frac{m}{2}[(1+2\lambda)\dot{q}_1^2 + \dot{q}_2^2].$$

$$\Rightarrow \begin{cases} \omega_1^2 = k\cdot\left(\frac{1}{m} + \frac{2}{M}\right) \\ \omega_2^2 = \frac{k}{m} \end{cases}$$

18.

non-linear oscillation

Nonlinear oscillations



$U(q) = \frac{1}{2}kx^2 + \frac{1}{3}\alpha x^3 + \frac{1}{4}\beta x^4$

use perturbation theory

$\ddot{x} + \frac{k}{m}x = -\frac{\alpha}{m}x^2 - \frac{\beta}{m}x^3 \stackrel{\Delta}{=} -\alpha x^2 - \beta x^3$

and write: $x = x^{(0)} + x^{(1)} + \cdots$ where $x^{(i)} \propto \alpha^{(i)}, \beta^{(j)}$

$$\begin{cases} \omega_0 = \sqrt{k/m} \\ x^{(0)} = A\cos(\omega_0 t) \quad \text{(reduce phase term)} \end{cases}$$

$\ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\frac{\alpha}{m}x^{(0)2} - \frac{\beta}{m}x^{(0)3}$

~~asymptotic perturbation theory~~ (slowly varying amplitude) $(\omega_0 t \ll 1)$

is there any orthogonal basis in this expansion? in function space?

like $\begin{cases} x^{(1)} = a\cos(\omega t) \\ \omega = \omega^{(0)} + \omega^{(1)} + \cdots \end{cases}$ $\quad \omega^{(i)} \propto \alpha^{(i)}, \beta^{(i)}$

and expand this to power of a. say.

$x^{(i)} \propto a^{(i)}$

rewrite Newton's Law:

$\frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3 + (1 - \frac{\omega_0^2}{\omega^2})\ddot{x}$

$$\Rightarrow \begin{cases} x^{(1)}: \frac{\omega_0^2}{\omega^2}\ddot{x}^{(1)} + \omega_0^2 x^{(1)} = 0 \\ x^{(2)}: \frac{\omega_0^2}{\omega^2}\ddot{x}^{(2)} + \omega_0^2 x^{(2)} = \alpha x^{(1)2} + (1 - \frac{\omega_0^2}{\omega^2})\ddot{x}^{(1)} \end{cases}$$

$= \alpha a^2 \frac{\cos 2\omega t + 1}{2} + (\omega^2 - \omega_0^2) a\cos\omega t$

notice that:

$\omega^2 - \omega_0^2 = \underbrace{2\omega_0\omega^{(1)}}_{\propto a^1} + \underbrace{2\omega\omega^{(2)} + \omega^{(1)2}}_{\propto a^2} + \cdots$

and ~~$x^{(2)}$~~ if $\omega^{(1)} = 0$:

$\frac{\omega_0^2}{\omega^2}\ddot{x}^{(2)} + \omega_0^2 x^{(2)} = \alpha a^2 \frac{\cos 2\omega t + 1}{2}$

$\Rightarrow x^{(2)} = -\frac{\alpha}{2}\frac{a^2}{\omega_0^2} + \frac{\alpha}{6}\cdot\frac{a^2}{\omega_0^2}\cos 2\omega t$

and for $x^{(3)}$:

$\frac{\omega_0^2}{\omega^2}\ddot{x}^{(3)} + \omega_0^2 x^{(3)} = -\beta a^3\cos^3\omega t - 2\alpha a\cos\omega t[-\frac{\alpha}{2}\frac{a^2}{\omega_0^2} + \frac{\alpha}{6}\frac{a^2}{\omega_0^2}\cos 2\omega t]$

$+ 2\omega^{(2)}\omega_0 a\cos\omega t.$ (not complete)

19.

!! the standard to do this:
① X_0 has form $|X_0| = a$ $\quad X_0 \propto$ ~~~~ X (origin form)
② expand X with a^n ③ should first evaluate the rank of ω related to a
④ $X^{(j)} \propto \cos(j\omega t)$

a small trick: Euler's equation to do $\cos^n$ or $\sin^n$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$\cos^2\theta = \frac{1}{4}(e^{2i\theta} + e^{-2i\theta} + 2) \qquad \sin^2\theta = -\frac{1}{4}(e^{2i\theta} + e^{-2i\theta} - 2)$$

$\vdots$

back to out $X^{(3)}$:

$$\frac{\omega_0^2}{\omega^2}\ddot{X}^{(3)} + \omega_0^2 X^{(3)} = -2\alpha X^{(1)}X^{(2)} - \beta X^{(1)3} - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{X} - 2\omega_0\frac{\omega^{(2)}}{\omega^2}X^{(1)}$$

$$\Rightarrow \quad X = -\cos 3\omega t\left[\frac{1}{6}\frac{\alpha^2 a^2}{\omega_0^2} + \frac{\beta a^3}{4}\right] + a\cos\omega t\left[2\omega_0\omega^{(2)}a - \frac{3\beta a^3}{4} - \frac{\alpha^2 a^3}{4}\left(-1 + \frac{1}{8}\right)\right]$$

$$\Rightarrow \quad \omega^{(2)} = \frac{a^2}{2\omega_0}\left[\frac{3\beta}{4} - \frac{5}{6}\frac{\alpha^2}{\omega_0^2}\right]$$

a thought:
$$\omega = \omega_0 + a\omega_1 + a^2\omega_2 + \cdots$$
$$X^{(1)} = a\cos\omega t = \mathrm{Re}(a e^{i\omega t}) = \mathrm{Re}(a e^{i\omega_0 t} \cdot e^{ia\omega_1 t} \cdot e^{ia^2\omega_2 t} \cdots)$$
and $e^{i a^n \omega_n t} = (1 + \Sigma_i (i a^n \omega_n t)^i / i!)$

back to see $\omega^{(2)}$ (this is a way to shift frequency !!)

for $\alpha > 0$, $\quad \omega^{(2)} < 0$ $\qquad T \uparrow$

for $\beta > 0$, $\quad \omega^{(2)} > 0$ $\qquad T \downarrow$

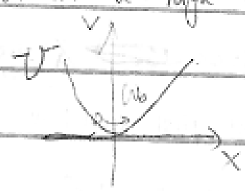
Could understand this by imaging the change of potential

$$\Rightarrow \quad X^{(3)} = \cos 3\omega t \cdot \frac{a^3}{16\omega_0^2}\left[\frac{\beta}{2} + \frac{\alpha^2}{3\omega_0^2}\right]$$

high frequency field.

(ok for $\int d\omega$)

motion in a high frequency field



$F(x,t) = f_1(x)\cos\omega t + f_2(x)\sin\omega t$.

where $\omega >> \omega_0$

method to this:

$X(t) = \underbrace{u(t)}_{\sim\omega_0} + \underbrace{\xi(t)}_{\sim\omega}$ + $m\ddot{x} = -\partial_x U + f(x,t)$

$|\xi| << u$

X(t):

$m\ddot{u} + m\ddot{\xi} \approx -\partial_x U|_{x=u} - \xi\partial_x^2 U + f + \xi\partial_x f$

for ξ: $m\ddot{\xi} = -\xi\partial_x^2 U + f$

for u: $m\ddot{u} = -\partial_u U + \xi\frac{\partial f}{\partial x}$ (notice $\xi\cdot\frac{\partial f}{\partial x} = \sim\omega^0 + \sim 2\omega$ then order of this is in u)

$\Longrightarrow$ for ξ: $\xi = -\frac{f}{m\omega^2}$. (a good physical interpretation: π phase change + not involved in $\omega_0(U)$)

for u: $m\ddot{u} = -\partial_u U - \underbrace{\frac{f}{m\omega^2}\frac{\partial f}{\partial x}} \approx \frac{1}{2m\omega^2}\partial_x f^2$ ← average over time is a good way

$= -\partial_u U - \frac{1}{4m\omega^2}\partial_x(f_1^2+f_2^2)$

$= -\partial_u U_{eff}$.

where $U_{eff} = U + \frac{1}{4m\omega^2}(f_1^2+f_2^2)$.

remark: ① compare this form with the ~~ex~~ derivation of external force in the beginning of the chapter. get: $\delta U = \frac{1}{4m\omega^2}(f_1^2+f_2^2)$ $= \frac{1}{2}m\dot{\xi}^2 \Rightarrow$ ~~kinet~~ E_T !!

② for arbitrary ~~ex~~ kinetic energy, $T = \frac{1}{2}a_{ij}\dot{q}_i\dot{q}_j$

then $\delta U = \overline{T}$

21.

parametric oscillation.

example.

$$\begin{cases} x = l\sin\theta \\ y = l\cos\theta - a\cos\omega t \end{cases}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$
$$= \frac{1}{2}m\left[l^2\dot\theta^2 - 2la\omega\sin\omega t\sin\theta\,\dot\theta + a^2\omega^2\sin^2\omega t\right]$$

$$\mathcal{L} = T + mgl\cos\theta$$
$$\sim \frac{1}{2}ml^2\dot\theta^2 - m l a\omega^2\cos\omega t\cos\theta + mg\cos\theta. \quad \text{(reduce a } \partial_t f \text{ term).}$$

(find f). $\Rightarrow \quad ml^2\ddot\theta = -mgl\sin\theta + mla\omega^2\cos\omega t\sin\theta$ (square average of this)

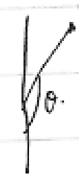
$$\Rightarrow \quad \delta U = m^2l^2a^2\omega^4\sin^2\theta / 4ml^2\omega^2 = (1/4)\,ma^2\omega^2\sin^2\theta$$

$$U = -mgl\cos\theta + \frac{1}{4}ma^2\omega^2\sin^2\theta$$

now think of $U'(\theta)=0$ point. say.

$$U' = 0 = mgl\sin\theta + \frac{1}{2}ma^2\omega^2\sin\theta\cos\theta.$$

$$\Rightarrow \quad \cos\theta_0 = -\frac{2gl}{a^2\omega^2} \quad (\leq 1).$$



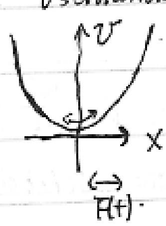
and for U'':

$$U'' = mgl\cos\theta + \frac{1}{2}ma^2\omega^2\cos 2\theta\,\Big|_{\theta=\theta_0} < 0. \quad \text{unstable.}$$

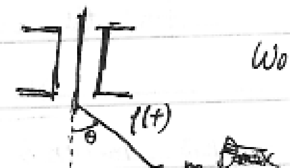
$$U''\big|_{\theta=0} > 0 \quad \text{stable}$$

$$U''\big|_{\theta=\pi} > 0 \quad \text{stable}$$

Parametric oscillations (resonance).

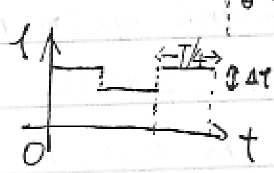


e.g.



$\omega_0 = \sqrt{g/l}.$

$\theta_{max} \to 0 \to \theta_{max} \to 0$



$$E = \frac{mv^2}{2} + mgh$$
$$W = (F_{max} - F_{min})\Delta l = \frac{3E}{l}\Delta l \quad !?$$
$$\frac{W}{T} = \frac{\delta E}{l}\Delta l \quad \Rightarrow \quad E \propto c\cdot\exp(-D\cdot t)$$

22.

~~think~~ think of dissipation force (friction).

$F_{fr} = 2\lambda\dot{x}$

$W_{fr} = \lambda\dot{x}^2$

$\Rightarrow \dot{E} = 2\lambda E / T \cdot m\omega_0^2 = E \cdot \frac{2\lambda}{Tm\omega_0^2}$ (not involving amplitude).

now think of the general cases:

$m\ddot{x} + kx = 0$

$\Rightarrow m(t), k(t)$ rewrite!!

$\Rightarrow \frac{d}{dt}(m\frac{d}{dt}x) + k(t)x = 0$

let $d\tau = \frac{1}{m(t)} dt$ ω^2

$\Rightarrow \frac{d^2}{d\tau^2}x + mkx = 0$

also, we ask periodic condition

$\omega_0 = \omega_0(t+T)$ $\nu = 1/T$ $(\omega = 2\pi/T)$

try to go back to the example in previous page

$a\cos\nu t$

$ml^2\ddot{\theta} = -mgl\sin\theta - mal\nu^2\cos\nu t\sin\theta$

$\Rightarrow \ddot{\theta} = -\theta \cdot \frac{1}{l}(g + a\nu^2\cos\nu t)$ ($\Leftrightarrow \ddot{x} + \omega_0^2(1 + u\cos\nu t)x = 0$)

perturbation theory:

$x = \sum p_n e^{in\nu t}$ $(*)$

find the resonance term. i.e.

$n\nu = \omega_0 - (-\omega_0)$ or $\nu = \frac{2\omega_0}{n} + \varepsilon$ (n odd ?)

remark: this motion's equation just like the Schrödinger equation with a periodic potential

for the case $n=1$, i.e. $\nu = 2\omega_0 + \varepsilon$

choose $(2j+1)\nu/2$ terms (because $\cos\nu t$ as $n\nu t$ ~~$\cos(\frac{n+1}{2})$~~)

say $x = a(t)\cos\frac{\nu}{2}t + b(t)\sin(\frac{\nu}{2})t$

$\dot{a}, \dot{b} \propto u \ll \omega_0$ $\quad \varepsilon \sim u\omega_0$ $(n \to n + 1/2)$

$$\ddot{x} \approx -\cancel{2\dot{a}(\ldots)} \quad -2\dot{a}\left(\tfrac{\nu}{2}\right)\sin\tfrac{\nu t}{2} - a\left(\tfrac{\nu}{2}\right)^2\cos\tfrac{\nu t}{2}$$
$$+2\dot{b}\left(\tfrac{\nu}{2}\right)\cos\tfrac{\nu t}{2} - b\left(\tfrac{\nu}{2}\right)^2\sin\tfrac{\nu t}{2}$$

now pay attention, we could have:

$$\left(\nu/2\right)^2 \approx \omega_0^2 + \omega_0\varepsilon \quad \tfrac{\nu t}{\varepsilon}$$

$$\Rightarrow -2\dot{a}\omega_0\sin(\nu t/2) - a\omega_0\varepsilon\cos\tfrac{\nu t}{2} + 2\dot{b}\omega_0\cos\tfrac{\nu t}{2} - b\omega_0\varepsilon\sin\tfrac{\nu t}{2}$$
$$= -\frac{\omega_0^2 u}{2}\left(a\cos\tfrac{\nu t}{2} \mp b\sin\tfrac{\nu t}{2}\right)$$

(this is the ~~form~~ form from (*)).

$$\Rightarrow \begin{cases} -2\dot{a} - b\varepsilon = \cancel{+}\,\frac{\omega_0 u}{2}\, b \\ -a\varepsilon + 2\dot{b} = -\frac{\omega_0 u}{2}\, a \end{cases}$$

(friendly to $a.b \propto e^{st}$)

$$\Rightarrow \quad s^2 = \left(\frac{\omega_0 u}{2}\right)^2 - \varepsilon^2$$

Interesting picture!

~~cos ... should be~~ $e^{i\omega_0 t}$ should be contained here.

What about ~~...~~ $k \cdot \frac{2\pi}{n} = \nu$?

it doesn't matter because the form

$$\ddot{x} + \omega_0^2(1 + u\cos\nu t) = 0 \text{ is self contained}$$

i.e.
$$\begin{cases} \omega_0 \to k\omega_0 \\ \nu \to k\nu \end{cases} \text{ is ok}$$

? QM?

yet if we are just interested in solving ε scale
could just fix the terms for $a(t)$, $b(t)$ to a, b.
$a(t)$, $b(t)$ here refers to resonance or damping

__dissipation.__

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2(1 + u\cos\nu t)x = 0.$$

Could use $x = e^{-\lambda t}x'$ to solve this

$\Rightarrow a, b \propto e^{st - \lambda t} \Rightarrow s > \lambda$ gives condition for resonance!

Floquet Theory:

$$\ddot{x} + \omega_0^2 x = 0 \qquad \omega_0 = \omega_0(t+T)$$

$x_1,\ x_2$ while

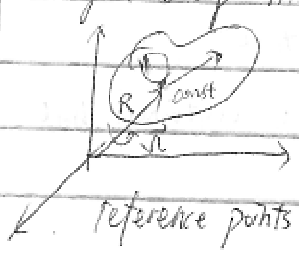
$$\begin{cases} x_1(t+T) = a x_1 + c x_2 \\ x_2(t+T) = b x_1 + d x_2 \end{cases}$$

$$\Rightarrow x_1\dot{x}_2 - x_2\dot{x}_1 = \text{const.} \quad \text{flux. (wronskian)}$$

$$\Rightarrow \begin{cases} x_1(t+T) = \mu_1 x_1 \\ x_2(t+T) = \mu_2 x_2 \end{cases} \qquad \begin{cases} x_1 = \mu_1^{t/T}\Pi_1 \\ x_2 = \mu_2^{t/T}\Pi_2 \end{cases}$$

$$\mu_1\mu_2 = 1, \quad \begin{cases} \mu_1 = e^{i\alpha} \quad \mu_2 = e^{-i\alpha} \\ \text{or } \mu_1 > 1 \quad \mu_2 < 1 \end{cases}$$

Ch6. Rigid body motion



$$\delta\vec{r} = \delta\vec{R} + \delta\vec{r}'$$

$$= \delta\vec{R} + \delta\vec{\varphi}\times(\vec{r} - \vec{R}) \quad (= \delta\vec{\varphi}\times\vec{r}'!)$$

$$\vec{v} = \vec{V} + \vec{\Omega}\times\vec{r}$$

change: $\vec{r} \Rightarrow \vec{r}' + \vec{a}$

$$\Rightarrow \vec{v}' = \vec{V}' + \vec{\Omega}\times\vec{r} \qquad \vec{V}' = \vec{V} + \vec{\Omega}\times\vec{a}$$

kinetic energy $= \frac{1}{2}\sum m v^2 = \frac{1}{2}\left(\sum m\vec{V}^2 + 2\vec{V}\sum m\vec{r} + \sum m(\vec{\Omega}\times\vec{r})^2\right)$

$$\overset{\text{com}}{=\!=} \frac{1}{2}M\vec{V}^2 + \frac{1}{2}\sum m(\vec{\Omega}^2\vec{r}^2 - \vec{\Omega}\cdot\vec{r}) \leftarrow T_{rot}$$

$$T_{rot} = \frac{1}{2}\sum m(\Omega^2 x_i^2 - \Omega_i x_i \Omega_j x_j)$$

$$= \Omega_i\Omega_j \frac{1}{2}\sum m(x_l^2\delta_{ij} - x_i x_j)$$

$$\triangleq \frac{1}{2} I_{ij}\Omega_i\Omega_j = \frac{1}{2}\Omega^T I\,\Omega$$

where $I = \sum m \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$ tensor of inertia

matrix $\rightarrow$ rotation transformation $\rightarrow$ we could do diagnalization

$$= \begin{pmatrix} X_2^2+X_1^2 & & \\ & X_1^2+X_3^2 & \\ & & X_2^2+X_1^2 \end{pmatrix}.$$

$I = \mathrm{diag}(I_1, I_2, I_3)$. $\quad I_1 + I_2 \geq I_3$ (when $=$??)

as we do a translation, say

$O \to O' + a$. then.

$I' = I + \cancel{\Sigma m} M(a^2 \delta_{ij} - a_i a_j)$

angular momentum :

$\vec{L} = \Sigma m \vec{r} \times \vec{v} = \Sigma m \vec{r} \times (\vec{V} + \vec{\Omega} \times \vec{r})$

$= M \vec{R} \times \vec{V} + \Sigma m (\vec{\Omega} \vec{r}^2 - \vec{r}(\vec{\Omega} \cdot \vec{r}))$.

$\vec{L}'$ = last term above

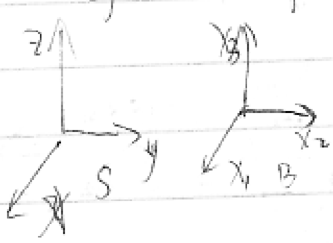
$= I_{ji} \Omega_j = I \vec{\Omega}$

Free rotation ~~free rotation~~.

$\vec{F} = 0$. $\vec{L} = $ const.

① spherical situation (top). $I_1 = I_2 = I_3 \equiv I$. $\vec{L} = I \vec{\Omega}$

② symmetric top $\quad I_1 = I_2 \neq I_3$ (choose this geometry !!)



if L = const in (X, Y, Z)

then $\left(\frac{d\vec{L}}{dt}\right)_S = \left(\frac{d\vec{L}}{dt}\right)_B + \vec{\Omega} \times \vec{L}$ $\Big(= 0$. stationary$\Big)$.

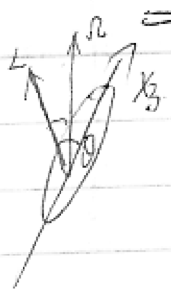
ω?

~~Free rotation~~

$\left(\frac{d}{dt}\vec{L}\right)_B = -\vec{\Omega} \times \vec{L}$

$\Rightarrow \begin{cases} \vec{L}^2 = \text{const}. \\ T_{rot} = \text{const}. = \Sigma \frac{L_i^2}{I_i} \end{cases} \qquad \vec{L} = \frac{\partial L}{\partial \vec{\theta}} \qquad (\vec{\theta} \text{ is } \frac{\vec{\Omega}}{4\pi} \cdots)$

since $I_1 = I_2 \neq I_3 \Rightarrow L_3^2 = $ const, $L_1^2 + L_2^2 = $ const



$\Rightarrow$ ~~and~~ $\Omega^2 = $ const.

$\vec{L}, \vec{\Omega}, \vec{X_3}$ in same plane.

~~precession~~ precession !! $\quad \begin{cases} \Omega_3 = \frac{L}{I_3} \cos\theta \\ \sqrt{\Omega_1^2 + \Omega_2^2} = \frac{L}{I_1} \sin\theta = \omega \sin\theta \end{cases}$

ω is precession ?

?

③ asymmetric top

$$I_1 < I_2 < I_3 \qquad \left(2E = \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right).$$

$$\Rightarrow 2EI_1 < L^2 < 2EI_3$$

External Force cases:

$$\begin{cases} P = MV \\ \dot{P} = F \end{cases}$$

$$\begin{cases} L = \Sigma m \vec{r} \times \vec{v} \\ \dot{L} = \Sigma \vec{r} \times \vec{F} \triangleq \vec{K} \end{cases} \Rightarrow \begin{cases} T_{rot} = \frac{1}{2} I_{ij}\Omega_i\Omega_j \\ \delta U(\varphi) = \vec{v} \times \vec{r} \cdot \delta\vec{\varphi} = \vec{K}\cdot\delta\vec{\varphi} \end{cases}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \vec{K} = \frac{dL}{dt} \quad ✓.$$

$$\Rightarrow d_t(\vec{L})_S = \vec{K} = d_t(\vec{L})_B + \vec{\Omega} \times \vec{L}.$$

$$(\vec{L})_B = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3)$$

$$\Rightarrow I_i\dot{\Omega}_i - \Omega_j\Omega_k\,\varepsilon_{ijk} I_j = K_i$$

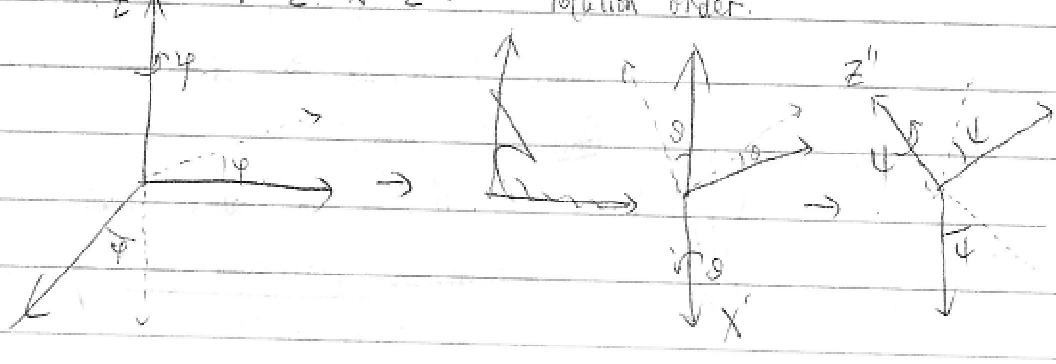
go back to the symmetry top. find: $(K_i = 0)$

$$\begin{cases} \Omega_3 = \text{const} \\ \dot{\Omega}_2 = \Omega_1\Omega_3 \frac{I_3 - I_1}{I_2} = -\omega\Omega_1 \\ \dot{\Omega}_1 = \Omega_2 \underbrace{\Omega_3 \frac{I_1 - I_3}{I_1}}_{\omega} = \omega\Omega_2 \end{cases} \Rightarrow \begin{cases} \Omega_1 = A\cos\omega t \\ \Omega_2 = -A\sin\omega t \end{cases}$$

$\Rightarrow \vec{L}$ is rotating! but not $\vec{K}$!!

Euler's angles:

$$\begin{cases} \varphi,\ \theta,\ \psi \\ Z\ \ X\ \ Z \end{cases} \quad \text{rotation order.}$$



$$\begin{cases} \vec{\Omega}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \vec{\Omega}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \vec{\Omega}_3 = \dot{\phi}\cos\theta + \dot{\psi} \end{cases} \qquad (\phi \leftrightarrow \varphi).$$

notice that : $\vec{\Omega}$ is a vector !! i.e. $\vec{\Omega}_1 + \vec{\Omega}_2 = \vec{\Omega}$!

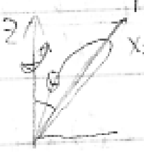
i.e. $d\vec{x}_1, d\vec{x}_2 \to \vec{\tau} + d\vec{\tau} \approx \vec{\tau} + d\vec{x}_1 \times \vec{\tau} + d\vec{x}_2 \times \vec{\tau}$. ✓

$\Rightarrow$ Euler's notation $\vec{\dot{\varphi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}} = \vec{\Omega}$

$\vec{\Omega} = \Omega_1 \vec{x}_1 + \Omega_2 \vec{x}_2 + \Omega_3 \vec{x}_3$ (projection).

e.g. symmetric top.

$$T_{tot} = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\varphi}\cos\theta)^2$$



$U = mgl\cos\theta$? (not used now)

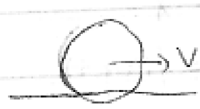
$L_z = \text{const} = \frac{\partial L}{\partial \dot{\varphi}} \qquad L_3 = \frac{\partial L}{\partial \dot{\psi}} = \text{const}$

(just like what we did in central force problem, think of full dimension and by calculating the conserved quantity, we get restriction ?)

$$? \begin{cases} \theta = \text{const} \,?\, \\ \dot{\varphi} = L/I_1 = (L_z - L_3\cos\theta)/I_1\sin^2\theta \\ \dot{\psi} = L_3/I_3 - L/I_1 \cos\theta = \Omega_3 \frac{I_1 - I_3}{I_1} \\ I_1\ddot{\theta} = I_1\dot{\varphi}^2\sin\theta\cos\theta + I_3(\dot{\psi} + \dot{\varphi}\cos\theta)\cdot(-\sin\theta)\dot{\varphi} \\ \quad = (L_z - L_3\cos\theta)^2/I_1\sin^3\theta \times \cos\theta - L_3\sin\theta \cdot L_z/I_1 \\ \quad = -\frac{\partial}{\partial\theta}\left[\frac{(L_z - L_3\cos\theta)^2}{2I_1\sin^2\theta}\right] \cdot \left(\stackrel{\Delta}{=} \partial_\theta U_\theta\right) \end{cases}$$

$\ddot{\theta} = 0 \Rightarrow L_z\cos\theta = L_3 \Rightarrow \dot{\theta} = 0$ and $L = L_z$.

non-holonomic systems : rigid body in contact.

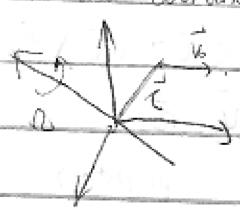


$(x, y, \theta, \varphi, \psi)$.

constrains $f(\{q_i, \dot{q}_i\}) = 0$

or $\sum C_i(q)\dot{q}_i = 0$

non-inertial coordinate frames



$\vec{V}_0 = \vec{u} + \vec{V} + \vec{\Omega} \times \vec{r}$ $\quad (\vec{u} = \dot{\vec{r}})$

what's U in this problem? what's L in this frame?

$$\vec{V}_0^2 = u^2 + V^2 + (\vec{\Omega} \times \vec{r})^2 + 2\vec{u}\cdot\vec{V} + 2\vec{u}\cdot(\vec{\Omega}\times\vec{r}) + 2\vec{V}\cdot(\vec{\Omega}\times\vec{r})$$

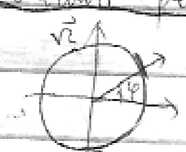
$$\Longrightarrow L = m\frac{u^2}{2} + \frac{m}{2}(\vec{\Omega}\times\vec{r})^2 - m\underbrace{\left(\frac{d\vec{V}}{dt} + \vec{V}\times\vec{\Omega}\right)}_{\triangleq W}\cdot\vec{r} + \vec{u}\cdot(\vec{\Omega}\times\vec{r}) - U(r)$$

$$\Longrightarrow \begin{cases} \frac{\partial L}{\partial \vec{u}} = m(\vec{u} + \vec{\Omega}\times\vec{r}) \\ \frac{\partial L}{\partial \vec{r}} = -\nabla U - mW + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega} + m\vec{u}\times\vec{\Omega} \end{cases}$$

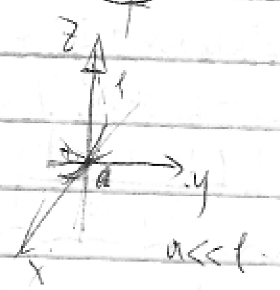
e.g. 1. $V = 0$. $\vec{\Omega} = \text{const}$.

$$m\dot{\vec{u}} = -\nabla U + 2m\vec{u}\times\vec{\Omega} + m(\vec{\Omega}\times\vec{r})\times\vec{\Omega}$$

e.g. 2. Foucault pendulum



$\Omega_\varphi = \Omega \sin\varphi$



$\alpha \ll 1$

reduce $(\vec{\Omega}\times\vec{r})\times\vec{\Omega} \approx \Omega^2 r$ ($u\Omega = \frac{\Omega r}{u}$)

$$m\ddot{\vec{r}} = -k\vec{r} + 2m\dot{\vec{r}}\times(\Omega_z \hat{z}).$$

$$\Longrightarrow \begin{cases} \ddot{x} + \omega^2 x = 2\dot{y}\Omega_\varphi \\ \ddot{y} + \omega^2 y = -2\dot{x}\Omega_\varphi \end{cases}$$

$$\xi = x + iy \Longrightarrow \ddot{\xi} + 2i\Omega_\varphi\dot{\xi} + \omega^2\xi = 0.$$

$(\Omega_\varphi \ll \omega)$

Ch 7 Hamiltonian equations of motion

$$\ddot{x} = f(x, \dot{x}, t) \Longleftrightarrow \begin{cases} \dot{y} = \dot{x} \\ \dot{y} = f(x, y, t) \end{cases}$$

$$dL = \dot{p}\,dq + p\,d\dot{q} = d(p\dot{q}) - \dot{q}\,dp + \dot{p}\,dq$$

let. $H = p\dot{q} - L$ then

$$dH = \dot{q}\,dp - \dot{p}\,dq \Longrightarrow \begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ -\dot{p} = \frac{\partial H}{\partial q} \end{cases}$$

intrinsic coordinate t?

possion brakets :

for $f = f(q, p, t)$.

$df = \partial_t f\, dt + \partial_p f\, dp + \partial_q f\, dq$

$= [\partial_t f \cancel{+} (\partial_p f \cdot (-\partial_q H) + \partial_q f (\partial_p H))]\, dt$

$\Rightarrow \begin{cases} \{f, g\} = \partial_p f\, \partial_q g - \partial_q f\, \partial_p g \\ df/dt = \partial_t f + \{H, f\}. \end{cases}$

it has the following properties :

① linearity

② $\{fh, g\} = f \cdot \{h, g\} + \{f, g\} \cdot h$

③ Jacobi's equation

$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

obviously :

① $d_t H = 0$.

② if $d_t f = 0$ $d_t g = 0$ then

$d_t \{f, g\} = 0$.

note : In fact here we first ① know L → ② know H

→ ③ write $\dot p, \dot q$ in p, q with the help of H. → ④ Solve for t.

now think of action S as a function of coordinates

$$S = \int_{(q_i, t_i)}^{(q_f, t_f)} L(q, \dot q, t)\, dt \xrightarrow{\delta S = 0} \underline{S(q, t)} \;!!!$$

$\dot q = \dot q(q, t)$

$\delta S = 0$ will just give us $\underline{1}$ path connecting initial and final state

$dS = L\, dt$.

idea : ① $\delta S = \int dt\, \delta q [\frac{d}{dt}(-\frac{\partial L}{\partial \dot q}) + \frac{\partial L}{\partial q})] + \frac{\partial L}{\partial \dot q} \delta q$

$\Longrightarrow \frac{\partial S}{\partial q} = p$

② $\frac{dS}{dt} = \frac{\partial S}{\partial q} \cdot \dot{q} + \frac{\partial S}{\partial t} = p\dot{q} + \frac{\partial S}{\partial t} = L.$

$\Rightarrow \frac{\partial S}{\partial t} = -H$

note: it might be better to decorate S with index 0. say S_0 to represent the chosen S that we are interested in.

③ combine ① + ②:

$dS = L\,dt = p\,dq - H\,dt.$

④. treat $\{p, q\}$ different, or unrelated then

$\delta S = \int \delta p\, dq + p\, d(\delta q) - (\partial_q H\, \delta q + \partial_p H\, \delta p)\, dt).$

$= \int [\, \delta p (dq - \frac{\partial H}{\partial p} dt) - \delta q (dp + \partial_q H\, dt)\,] + p\,\delta q.$

$\Rightarrow \dot{q} = \partial_p H \quad -\dot{p} = \partial_q H$

thus we could say, equation in ③ could help develop everything that we want in ~~the~~ Hamiltonian mechanism with the help of $\delta S = 0$!!

Maupertuis principle (forget about Lagrange first).

trajectory, time independent system $H(p, q) = \text{const}.$

$\delta q = 0$. yet $\delta t \neq 0$

$\Rightarrow E = \text{const}$ gives me trajectory

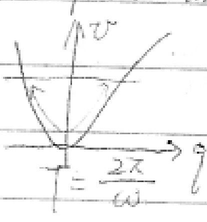
$E = V + \frac{1}{2} a\, \dot{q}_i \dot{q}_j \Rightarrow dt = \left(\frac{a\, dq\, dq}{2(E - V)} \right)^{1/2}$

insert to $\delta S_1 = 0 = \delta \int p\, dq = 0$? wtf?

adiabatic invariants

$$\begin{cases} S = S_0 - E(t - t_i) \\ \delta S + E\,\delta t = 0 \end{cases} \Rightarrow t - t_i = \partial_E S_0$$

note that here we use a "modified" $\delta S = 0$. i.e. no longer fix the final time but set it to be time-dependent.



$H = (p, q, \lambda)$

$\lambda(t) \longrightarrow \dot{\lambda} T \ll \lambda$

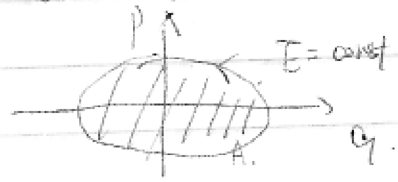
$d_t E = d_t \lambda \cdot \partial_\lambda H = d_t \lambda \cdot \overline{\partial_\lambda H} = d_t \lambda \cdot \frac{1}{T} \int \partial_\lambda H\, dt$

$$= d_t\lambda \cdot \frac{1}{T} \oint \frac{dq}{\partial_p H} \cdot \partial_\lambda H$$

$$\frac{H = E = \text{const}}{T = \oint dq\, \partial_p H} \quad d_t\lambda \; \frac{\oint dq \left(-\frac{dp}{d\lambda}\right)}{\oint dq : \partial_p H} \Longrightarrow \frac{d}{dt}\left(\oint p\,dq\right) = 0$$

i.e. $S_0 = \oint p\,dq = \text{const} \triangleq 2\pi I.$

$$\partial_E I = T/2\pi \qquad \cancel{\partial_\lambda I} \; \cancel{\partial I}\, \partial_I E = \omega. \quad (E = \hbar\omega!!)$$



$$\oint p\,dq = \iint dp\,dq = S_A$$

Canonical transformation. (symplectic geometry)

$$(q, p) \longleftrightarrow (Q, P)$$

$$\begin{cases} \delta S = 0 & \text{(give us Hamiltonian equation)} \\ dS - dS' = dF \end{cases}$$

Variables come out as pairs i.e. $(q, \cancel{p}Q)$, (Q, p), (q, P), $(\cancel{Q}p, P)$

we have 1 ~~~~ large freedom. i.e. ~~F is random~~

$\Rightarrow$ if $F = F(q, Q, t)$ then.

$$p = \partial_q F \quad -P = \partial_Q F \quad \partial_t F = H' - H$$

interesting part is (canonical)

$$\partial_t F = 0. \quad \text{then}$$

$$\eta \triangleq \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \qquad \{\eta^T, \eta\}_{Pb} = \begin{bmatrix} & I \\ -I & \end{bmatrix} = \{\eta'^T, \eta'\}_{Pb}$$

and. $\int dq\,dp = \int dQ\,dP$

Some notes on homework.

HW 2.1.

after writing L, could try to diagonalize it. (to give oscillation)

$a = \left[\frac{m_2}{m_1+m_2} l_1 l_2\right]^{1/2}$ $b = \frac{l_1-l_2}{2}$ $\lambda = \frac{1}{a}\left[(b^2+a^2)^{1/2} - b\right]$.

$\Omega^2 = g / \left[\frac{l_1+l_2}{2} \pm (a^2+b^2)^{1/2}\right]$ $b^2+a^2 = \left(\frac{l_1+l_2}{2}\right)^2 - \cancel{4} l_1 l_2 + \frac{m_2}{(m_1+m_2)} l_1 l_2$

$= \left(\frac{l_1+l_2}{2}\right)^2 + l_1 l_2 \cdot \frac{m_1}{m_1+m_2}$

but this is equal to solve it.

HW. 4.1.

a good integration:

$$\int_0^{v} \frac{dE}{\sqrt{E(v-E)}} = \pi.$$

~~HW 5.1~~.

HW 6.2. (a) $\frac{dV}{dr} = 0. \Rightarrow \frac{dV}{dr}\Big|_{r=R} = \frac{l^2}{\mu R^3}$ (b) $\Rightarrow \omega = \left(\frac{V''}{\mu}\right)^{1/2}$

$\Rightarrow \Omega = \frac{l}{\mu R^2} \Rightarrow \phi_0 = \frac{\pi \Omega}{\omega} = \pi \cdot \left(\frac{dU/dr}{3\, dU/dr + r\, d^2U/dr^2}\right)^{1/2}\Big|_{r=R}$

(c). $U' = c(3U' + rU'') \Rightarrow U = \frac{a}{\lambda+1} r^{\lambda+1}$ $(\lambda = \frac{1}{c} - 3)$. $(c>0)$.

(d). for $\phi_0 = \pi \cdot \frac{m}{n}$. $\Rightarrow \phi_0 = \cdots$. $\phi_0 = \frac{\pi}{(2+\alpha)^{1/2}}$

HW 9.1 how to calculate eigen vector

$A = \begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}$. $(A-\lambda I)X = \begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \overset{\text{some how}}{=} \begin{pmatrix} \vdots & \\ 0 & \\ 0 & \end{pmatrix} X = \begin{pmatrix} 0 \\ 0\ 0 \\ 0 \end{pmatrix} X = \cdots$

$= \begin{pmatrix} 0 & \\ 0\ 0 & pqr \\ 0 & 0\ 0 \end{pmatrix} X \Rightarrow x_1, x_2, \cdots x_{i-1} = 0$ is eigenvector

$\underbrace{p x_i + q x_{i+1} + r x_{i+2} + \cdots}_{\triangleq \alpha} = 0$.

$\Rightarrow A\alpha = \lambda\alpha$. !!

prove to Bertrand Theorem.
most important difference is : talk until V'', V'''.

① energy ~~centery~~ conservation :

$$E = \frac{m}{2}\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r).$$

$$\Rightarrow m\ddot{r} = \frac{\ell^2}{mr^3} - \frac{dV}{dr}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}\frac{d}{du}V(1/u).$$

$$\triangleq J(u) \triangleq -\frac{m}{L^2u^2}f(1/u). \quad (f(r) = V'(r)).$$

② circle road : $u_0 = J(u_0) = -\frac{m}{L^2u_0^2}\cdot f(1/u_0)$

③ oscillation : $y = u - u_0$

$$J(u) = J(u_0) + y J_0' + \frac{1}{2}y^2 J_0'' + \frac{1}{6}y^3 J_0'''$$

$$\beta^2 = 1 - J_0'$$

$$(\ast) \quad \Rightarrow \frac{d^2y}{d\theta^2} + \beta^2 y = \frac{1}{2}y^2 J_0'' + \frac{1}{6}y^3 J_0'''$$

and $y = h_1\cos(\beta\theta)$.

④ make use of u_0 :

$$J'(u_0) = J'(u)\big|_{u=u_0} = 1 - \beta^2 = -2 + \frac{u_0}{f(1/u_0)}\cdot\frac{df}{du}$$

$$\Leftrightarrow \frac{df}{dr}\Big|_{r_0} = (\beta^2 - 3)\frac{f}{r}\Big|_{r_0}$$

⑤ make use of the condition that $\Delta\phi$ not vary with R.

$$\frac{df}{dr} = (\beta^2 - 3)\frac{f}{r}.$$

$$\Rightarrow f \propto -u^{3-\beta^2}$$

⑥ Fourier expansion.

$y(\theta) = h_0 + h_1\cos\beta\theta + h_2\cos 2\beta\theta + h_3\cos 3\beta\theta$. and back to $(\ast)$.

$$\Rightarrow h_0 = h_1^2 \cdot J_0''/4\beta^2$$

$$h_2 = -h_1^2 J_0''/12\beta^2$$

$$h_3 = -\frac{1}{8\beta^2}\left[h_1 h_2\frac{J_0''}{2} + h_1^3\frac{J_0'''}{24}\right]$$

tip. h 留 h_1^2 留 $h_1 \cdot h_i$ 留 3阶只留 h_1^3. say:

$$\beta^2 \eta = \beta^2[h_0 + h_1 \cos\beta + h_2 \cos 2\beta + h_3 \cos 3\beta]$$

$$\eta'' = -\beta^2[h_1 \cos\beta + 4h_2 \cos 2\beta + 9 h_3 \cos 3\beta].$$

$$\frac{1}{2}J_0''\eta^2 = \frac{1}{4}J_0''[h_1^2 \cdot \frac{1}{2} + 2(2h_1h_0 + h_1h_2)\cos\beta + \cos 2\beta \cdot h_1^2]$$

$$\frac{1}{6}J_0'''\eta^3 = \frac{1}{6}J_0''' \cdot h_1^3 \cos^3\beta$$

$$= \frac{1}{24}J_0''' \cdot h_1^3 (\cos 3\beta + 3\cos\beta).$$

先看 $\cos^0$, $\cos^2$, $\cos^3$ 再看 $\cos^1$.

$$\Rightarrow 0 = (2h_1h_0 + h_1h_2)\frac{J_0''}{2} + h_1^3 \cdot \frac{J_0'''}{8} = \frac{h_1^3}{24\beta^2} \cdot [3\beta^2 J_0''' + 5(J_0'')^2]$$

$$\Rightarrow \beta^2(1-\beta^2)(4-\beta^2) = 0.$$

and for the angel. we have:

$$\eta(\theta) = h_1 \cos(\beta\theta).$$

thus it refers to $\theta_0 = \frac{2\pi}{\beta} = \begin{cases} \infty. & \beta = 0. \quad \text{perfect circle} \\ \pi. & \beta = 2 \quad \text{harmonic} \\ 2\pi. & \beta = 1. \quad \text{gravity} \end{cases}$

$$V \propto \int f\,dr \propto -\int r^{\beta^2-3}dr \propto r^{\beta^2-2} \begin{cases} r^2 & \text{harmonic} \\ r^{-1} & \text{gravity.} \\ r^{-2} & \text{perfect} \end{cases}$$

CH3. Integration of the equation of motion

1-dim motion

$$\begin{cases} L = \frac{1}{2}a\dot{q}^2 - U(q). \\ t = \sqrt{\frac{m}{2}}\int \frac{dx}{\sqrt{E-U}} \end{cases}$$

period and potential

$$X_2(U) - X_1(U) = \frac{1}{\sqrt{2m}\,\pi}\int_0^U T(E)\cdot\frac{dE}{\sqrt{U-E}}$$

2-body system

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - U(|\vec{r}_1-\vec{r}_2|).$$

$$\begin{cases} m\vec{r}_1 + m\vec{r}_2 = \text{const} \\ L' = \frac{1}{2}\mu\dot{r}^2 - U(r). \end{cases}$$

Central force problem

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

$$\begin{cases} l = mr^2\dot{\phi} = \text{constant}. \\ U_{eff} = U(r) + l^2/2mr^2 \\ \dot{r} = \left[\frac{2}{m}(E-U) - \frac{l^2}{m^2r^2}\right]^{1/2} \\ \phi = \int \frac{(l/r^2)\,dr}{[2m(E-U) - (l^2/r^2)]^{1/2}} \end{cases}$$

when to reach $r=0$.

$$\begin{cases} E_T = E - U - l^2/2mr^2 > 0 \\ Ur^2 < -l^2/2m. \end{cases}$$

closed loop problem

$$\Delta\phi\Big|_{r_{min}}^{r_{max}} = 2\pi\cdot\frac{m}{n}$$

Kepler's problem

$$U = -\alpha/r.$$

$$\phi = \arccos\frac{l/r - \frac{m\alpha}{l}}{\left[2mE + \frac{m^2\alpha^2}{l^2}\right]^{1/2}} + C.$$

$$\begin{cases} p/r = 1 + e\cos\phi. \\ a = \frac{\alpha}{2|E|}. \\ b = l/\sqrt{2m|E|} \\ T = \pi\alpha\sqrt{m/2|E|^3} \\ r = a(1-e\cos\xi) \\ t = \sqrt{ma^3/\alpha}\,(\xi - e\sin\xi) \end{cases}$$

$$\begin{cases} x = a(\cos\xi - e) \\ y = a\sqrt{1-e^2}\sin\xi. \end{cases}$$

tricks: 1. $K(k) = \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1-k^2\sin^2\xi}}$

2. $r - a = -ae\cos\xi$

3. $\Gamma(t) = \int_0^{+\infty} x^{t-1}e^{-x}dx.$

4. $\Delta\phi = -2\frac{\partial}{\partial l}\int_{r_{min}}^{r_{max}} \left[2m(E-U) - \frac{l^2}{r^2}\right]dr$

(Hamiltonian equation is easy).

5. $\delta\phi = \frac{\partial}{\partial l}\left[\frac{2m}{l}\int_0^{\pi} r^2\,\delta U\,d\phi\right].$

CH4.

Collisions between particles.

$\downarrow$

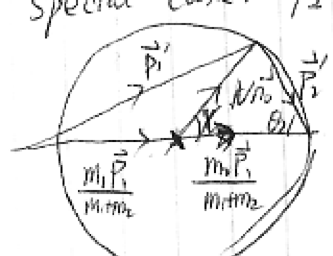
geometry of the collisions.

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = \vec{P}'$$

elastic
$$\vec{p}_1' = m\vec{v} + \frac{m_1}{m_1+m_2}\vec{P}$$
$$\vec{p}_2' = -m\vec{v} + \frac{m_2}{m_1+m_2}\vec{P} \quad \begin{cases} m = \frac{m_1 m_2}{m_1+m_2} \\ \vec{v} = \vec{v}_1 - \vec{v}_2 \end{cases}$$

$\downarrow$

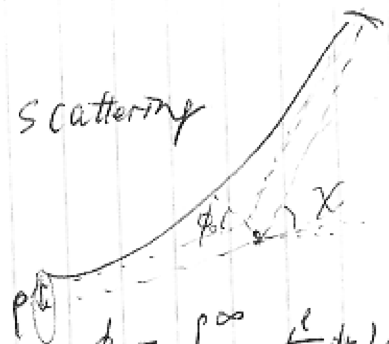
a special case. $\vec{p}_2 = 0$



χ vs. θ_2

change of angle of $\vec{p}_2$!!

$\longrightarrow$ scattering



$E = \frac{1}{2} m v_\infty^2$

$\ell = m \rho v_\infty$

$$\phi_0 = \int_{r_{min}}^{\infty} \left(\frac{\ell}{r^2} dr\right) \Big/ \left[2m(E-U) - \ell^2/r^2\right]^{1/2}$$

$\rightarrow$ "fall down" problem.

$E > U_{eff,\max}$

$$d\sigma = 2\pi \rho \, d\rho$$
$$d\Omega = 2\pi \sin\chi \, d\chi$$

$\longrightarrow$ rigid body $\rho = a\cos\frac{1}{2}\chi$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}a^2 \frac{1+(m_1/m_2)^2\cos 2\theta_1}{[1-(m_1/m_2)^2 \sin^2\theta_1]}.$$

$\downarrow$

Use $\frac{d\sigma}{d\Omega}$ to deduce $U(r)$. (has some requirements.

$$W = (1 - U/E)^{1/2}$$
$$= \exp\left[-\frac{1}{\pi}\int_{rw}^{\infty} \frac{\chi(\rho)\, d\rho}{\sqrt{\rho^2 - r^2 w^2}}\right] !?$$

potential well

$$\begin{cases} n = \sqrt{1 + 2U_0/mv_\infty^2} \\ \frac{\sin(\alpha - \frac{1}{2}\chi)}{\sin\alpha} = \frac{1}{n} \end{cases}$$

$$\frac{d\sigma}{d\Omega} = a^2 \frac{n^2}{4\cos\frac{\chi}{2}} \frac{(n\cos\frac{\chi}{2} - 1)(n - \cos\frac{\chi}{2})}{(n^2 - 2n\cos\frac{\chi}{2} + 1)^2}$$

Rutherford scattering

$$U = \alpha / r.$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{2mv_\infty^2}\right)^2 \Big/ \sin^4\frac{1}{2}\chi$$

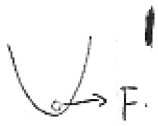
Some approximations here $\begin{cases} \text{geo} \\ \text{quantie} \end{cases}$

trick: small-angle scattering:

$p_{1y}' \ll p_{1x}' \approx p_{1x}$ $\quad \theta_1 \approx p_{1y}' / m_1 v_\infty = \int_{-\infty}^{+\infty} F_y dt / m_1 v_\infty$

$$F_y = -\frac{dU}{dr}\frac{y}{r} \approx -\frac{dU}{dr}\frac{\rho}{r}. \quad dt \approx \frac{dx}{v_\infty}.$$
$$r = \sqrt{\rho^2 + x^2}.$$

Ch 5. Oscillations

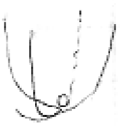


1. forced $L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2+xF(t)$ $\ddot{x}+\omega^2x=F(t)/m$

$\xi=\dot{x}+i\omega x$ $E=\frac{1}{2}m|\xi|^2=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$

$=F(t)*e^{i\omega t}/m$ $(=U+\frac{F^2}{2m\omega^2}=\frac{1}{2m\omega^2}|\int_{-\infty}^{+\infty}\dot{F}(t)e^{i\omega t}dt|^2).?$

2. multi-degree of freedom.



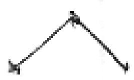
$$L=\frac{1}{2}(\dot{x}^TM\dot{x}-x^TKx).$$

diagnalize it $\rightarrow$ $|K-\omega^2M|=0.$ $x=\alpha e^{i\omega t}.$

$\lambda=A^TKA \Rightarrow \cancel{\lambda=\omega^2A^TMA} + A^TMA=I.$

3. modes:

① total 3n, 3 translation, 3 rotation, 3n−6 virbration



②. collinear: 3T, 2R, 3n−5 V.

③ on plane: 2n total, 2T, 1R, 2n−3 V, n−3 V out of plane

④ plane + linear: n total, 1T, 0R, n−1 V, 2n−4 V out of plane (yet n−2 ω).

4. damped oscillation

① L = ? $m\ddot{x}+\alpha\dot{x}+kx=0$

or $\ddot{x}+2\lambda\dot{x}+\omega_0^2x=0$

② L = ? $M\ddot{X}+KX=-\alpha\dot{X}$

$F=\frac{1}{2}\dot{x}^T\alpha\dot{x}$ $\frac{d}{dt}p_i=\frac{\partial L}{\partial x_i}-\frac{\partial F}{\partial\dot{x}_i}$ $\frac{dE}{dt}=-\dot{x}_i\frac{\partial F}{\partial\dot{x}_i}=-2F_{fric}$

5. forced + friction

$$\ddot{x}+2\lambda\dot{x}+\omega_0^2x=f/m.$$

green function method.

$\ddot{g}+2\lambda\dot{g}+\omega_0^2g=\delta/m$ $\rightarrow$ Fourier transform

$x=f(t)*g(t).$

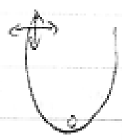
usually need the contour integration!

$$g(t)=e^{-\lambda t}\cdot\frac{-i}{m}\cdot\frac{1}{2\bar{\omega}}[e^{i\bar{\omega}t}-e^{-i\bar{\omega}t}]=\frac{e^{-\lambda t}}{m}\cdot\frac{1}{\bar{\omega}}\sin\bar{\omega}t$$

$$\bar{\omega}=\sqrt{\omega_0^2-\lambda^2}$$

$I \stackrel{A}{=} -\frac{dE}{dt}=-2F_{fric}$

note: if f is simple. e.g. $f=e^{i\alpha t}+e^{i\beta t}$ then just use $x=Ae^{i\alpha t}$ is easy



6. parametric resonance:

$$\ddot{x} + \omega^2(t)\, x = 0.$$

$$\omega^2 = \omega_0^2 (1 + h\cos\nu t).$$ ν is around some $\frac{2\omega_0}{n}$. $\triangleq \frac{2\omega_0}{n} + \varepsilon$.

n odd, choose $x = a_1 \cos(\frac{\nu}{2}t) + a_2 \cos(\frac{3}{2}\nu t) + \cdots$
$+ b_1 \sin(\frac{\nu}{2}t) + b_2 \sin(\frac{3}{2}\nu t) + \cdots$

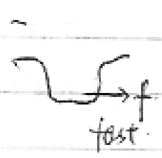
n even, choose $x = C + a_1 \cos\nu t + a_2 \cos 2\nu t + \cdots$
$+ b_1 \sin\nu t + b_2 \sin 2\nu t + \cdots$

$\Rightarrow$ give us ε.

① use matrix to write the equation.

② $a, b \sim \exp(st)$ yet if we set $s=0$. could find $|\varepsilon_{max}|$.

③ for any n, $|\varepsilon_{max}| \sim h^n$ $\quad \varepsilon_{max} \sim h^{n-1} \pm h^n$.

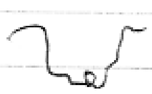


7. random potential + high frequency field

$$m\ddot{x} = -\partial_x U + f. \qquad f \sim e^{i\omega t}.$$

$x(t) = X(t) + \xi(t)$. $\quad \xi \ll X$. average. Taylor expansion.

$$\Rightarrow U_{eff} = U + \tfrac{1}{2} m \overline{\dot{\xi}^2} \qquad \xi = -f/m\omega^2 \;!!$$



8. anharmonic potential oscillation

$$m\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3$$

$x^{(1)} = a\cos\omega t.$ $\quad \omega = \omega_0 + \omega_1 + \cdots \quad \omega_i \sim a^i$

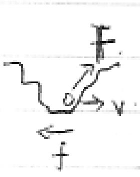
$x^{(2)}, x^{(3)} \cdots$ does not contain $\cos\omega t$ term!

① order of a. ② cancel order of $\cos\omega t$.

$$\frac{\omega_0^2}{\omega^2}\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\ddot{x}$$

$$x^{(2)} = -\alpha \frac{a^2}{2\omega_0^2} + \alpha \frac{a^2}{6\omega_0^2}\cos 2\omega t \qquad \omega_1 = 0.$$

$$x^{(3)} = \frac{a^3}{16\omega_0^2}\left(\frac{\alpha^2}{3\omega_0^2} - \frac{1}{2}\beta\right) \qquad \omega_2 = \left(\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3}\right) a^2$$



9. sum over all

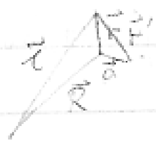
$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = f - \alpha x^2 - \beta x^3 \quad \text{how to ?}$$

Ch6.

rigid body / rotation motion (lack tricks).

motion. representation with rotation.

1. what is rotation in rigid body. and angular velocity



$d\vec{r} = d\vec{r} + d\vec{R} = d\vec{a} + d\vec{r}' + d\vec{R}$

rotation around ($\vec{r}$) rotation around ($\vec{r}'$)

$\Rightarrow \vec{\Omega}' = \vec{\Omega}$ and $\vec{V}' = \vec{V} + \vec{\Omega} \times \vec{a}$. (rather then $+\vec{a}$).

2. and instantaneous axis

Lagrangian

2. the kinetic energy and Lagrangian Coordinate!!

$T_K = \frac{1}{2}\mu V^2 + \frac{1}{2}\sum m(\vec{\Omega}\times\vec{r})^2 = T_K + T_{rot}$

rotation axis ~~around~~ pass cms

$L = T - U$ $T_{rot} = \frac{1}{2}\Omega_i I_{ik}\Omega_k$ $I_{ik} = \int d\rho (x_l^2\delta_{ik} - x_i x_k) dV$.

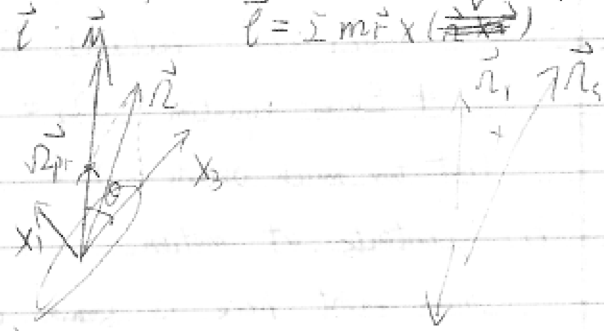
symmetric top $I_1 = I_2 \neq I_3$ asymmetric top $I_1 \neq I_2 \neq I_3$ sphere top

rotation of axis

3 angular momentum $\vec{l}$ $\vec{M}$ $\vec{l} = \sum m\vec{r} \times (\vec{v})$ (cms?)

$\vec{l} = I\vec{\Omega}$

$I_1 = I_2 \neq I_3$



if we choose $T_{rot} = \sum \frac{1}{2}\Omega_i I_i \Omega_i$ then the axis also rotates!

// then we could say $\vec{M}$ is fixed yet $\vec{\Omega}$ is not fixed. and $\vec{\Omega}_i$ is fixed

∴ the next step is how to describe these $\vec{e}_i$

this rotation ~~could~~ of the axis is easy to make out. Say $\Omega_{pr} = \frac{\vec{M}}{I_1}$

Conserved quantity?

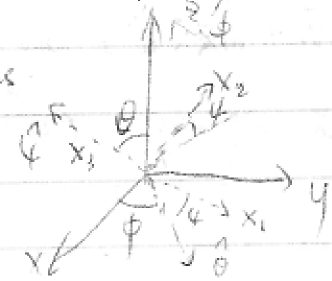
4 external force

$\frac{d\vec{l}}{dt} = \vec{K} = \sum \vec{r}\times\vec{f}$ in cms frame !!!

or $\frac{d}{dt}\left(\frac{\partial L}{\partial \Omega}\right) = \frac{\partial L}{\partial \phi}$

repentation of rotation of axis

I. Eular angles



$\vec{\Omega} = \dot{\vec{\psi}} + \dot{\vec{\theta}} + \dot{\vec{\phi}}$

this angular ~~~~ velocity is just the one we talked about. they are fixed yet $\vec{x}_1, \vec{x}_2, \vec{x}_3$ changes

Solution: find $\vec{\Omega}$. use $T = T_c + T_{rot}$

where $T_{rot} = \frac{1}{2}\vec{\Omega}^T I \vec{\Omega}$ and $\vec{\Omega}$ does not rely on which point you choose.

then it's good to chose the axis of spontaneous axis. $\vec{V} = 0$.

~~usually that is the contact point.~~ (line)

6. Euler's equation

differential equation if external force

① general case. $\frac{d\vec{A}}{dt} = \vec{\Omega} \times \vec{A} + \frac{d\vec{A}'}{dt}$ = change with "axis" + change compared to frame

② now use $\vec{A} = \vec{P}$ and $\vec{A} = \vec{l}$.

③ use $\frac{d\vec{l}}{dt} + \vec{\Omega} \times \vec{l} = \vec{K}$. get:

$$I_i \dot{\Omega}_i + (\Omega_j \Omega_k / I_i) \varepsilon_{ijk} \frac{1}{2} (I_k - I_j) = K_i$$

In the rotating frame passing one!

7. solve for asymmetric top!!

Convervation law (two integration constant)

$$2E = \vec{\Omega}^T I \vec{\Omega} \qquad M^2 = M_i^2$$

because for symmetric top it's easy to find M_i = const!

$$\Omega_1, \Omega_3 = \Omega_1(\Omega_2), \Omega_3(\Omega_2).$$

+ Euler equation $\dot{\Omega}_2 = \dot{\Omega}_2(\Omega_1, \Omega_3) = \dot{\Omega}_2'(\Omega_2)$. ✓

(not too much details ...).

8. contacts of motion.

main point here is some constraints that could not be expressed as $\lambda(A-B) = 0$. then add to Lagrangian $L + \lambda(A-B) = 0$

i.e. $\Sigma C_{ij} \dot{q}_i = 0$. could not find to be total derivative

then solution is just:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \lambda_i C_{ij}$$

9. motion in a non-inertial frame

$$K_0 \xrightarrow{\text{translation}} K' \xrightarrow{\text{rotate}} K$$

$L(v) \Rightarrow L'(v') \Rightarrow L(v)$ and reduce the $\frac{df}{dt}$ in L.

$$\Rightarrow m\ddot{\vec{x}} = -\nabla U - m\vec{W} + m\vec{x} \times \dot{\vec{\Omega}} + 2m\vec{v} \times \vec{\Omega} + m\vec{\Omega} \times (\vec{r} \times \vec{\Omega})$$

translational un-uniformity Corriolis Centrifugal.

Ch 7. canonical equations

① Hamilton equation

$L \rightarrow p\dot{q} - L = H(q,p,t)$.
(not use notion right now)

Routhian equation

$L \rightarrow p_i \dot{q}_i - L = R(q_i, p_i, q_j, \dot{q}_j, t)$

② possion brakets.

$[f,g] = \partial_p f \partial_q g - \partial_p g \partial_q f$

$d_t f = \partial_t f + [H, f]$.

make use of motion already

possion's theorem.

Liouville theorem

⑤ adiabatic invariants

$I = \oint p\,dq / 2\pi$.

★ canonical variables

\+ accuracy of them

④ canonical transformation

$L_1 - L_2 = F$

$(p,q) \rightarrow (P,Q)$

③ action $S = S(q,t)$

fix q and t, solve $\delta S = 0$, get S finally.

$dS/dt = L'(q,t) = L(q, \dot{q}(q,t), t)$.

$dS = p\,dq - H\,dt$. $S = \int \cdots$

could rewrite the whole equation by ignore the relation before.

once we know $H(q,p,t)$. make an attention on what coordinate we are using;

H–J equation.

$\frac{\partial S}{\partial t} + H(p,q,t) = 0$

$\Updownarrow$

$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0$.

here because we are trying to solve sth. rather than Lagragian

Solve it, seperate variable tricks like schrödinger equation

$F = \frac{\partial S}{\partial t} \Rightarrow \alpha_i, \beta_i = \text{const}$!!

Maupertuis principle.

just now consider δq. right now consider δt

$\delta S_0 = 0 \Rightarrow$ conserved energy path

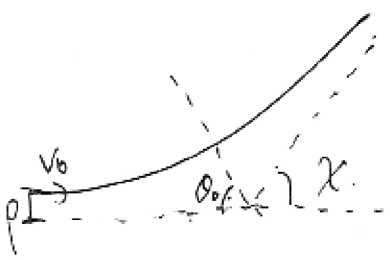
$\Rightarrow$ main aim is find expression for ~~δS_0~~

$S_0 \Rightarrow$ could also tell the time

$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$.

$\Rightarrow \begin{cases} l = mr^2\dot{\phi} \\ E = \frac{1}{2}m(\dot{r}^2) + \frac{l^2}{2mr^2} + V(r). \end{cases}$ $\quad H = \frac{1}{2m}(p_r^2 + p_\theta^2/r^2) + V(r)$.

$$\left.\begin{aligned} \frac{dr}{dt} &= v_r = \sqrt{\frac{2}{m}} \cdot \sqrt{E - V - \frac{l^2}{2mr^2}} \\ \frac{d\phi}{dt} &= \frac{l}{mr^2} \end{aligned}\right\} \quad \begin{aligned} \frac{d\phi}{dr} &= \frac{l}{\sqrt{2m}} \frac{1/r^2}{\sqrt{E - V - \frac{l^2}{2mr^2}}} \\ &= -\frac{\partial}{\partial l}\left(\sqrt{E - V - \frac{l^2}{2mr^2}} \cdot \frac{1}{\sqrt{2m}} 2m\right). \\ &= \sqrt{2m}\, \frac{\partial}{\partial l}\left(\sqrt{E - V - \frac{l^2}{2mr^2}}\right). \end{aligned}$$



$r \to +\infty$. $\theta \to \theta_0$. $\chi = \pi - 2\theta_0$.

$$\theta_0 = \int_{r_{min}}^{+\infty} \sqrt{2}\sqrt{m}\, \frac{\partial}{\partial l}\left(\sqrt{E - V - \frac{l^2}{2mr^2}}\, dr\right).$$

$$= \sqrt{2m}\, \frac{\partial}{\partial l}\left(\int_{r_{min}}^{+\infty} \sqrt{E - V - \frac{l^2}{2mr^2}}\, dr\right).$$ (refers to $l = p_\theta$, $\dot{\theta} = \frac{\partial H}{\partial p_\theta}$).

$$\delta\theta = \sqrt{2m}\, \frac{\partial}{\partial l}\left(\int_{r_{min}}^{+\infty} \frac{\delta V}{\sqrt{E - V - \frac{l^2}{2mr^2}}}\, dr\right).$$

$$\approx 2m \frac{\partial}{\partial l}\left(\int_0^{\theta_0} \delta V \cdot r^2\, d\theta\right).$$ amazing outcome. right?

r_{min} equation: $\begin{cases} E = \frac{l^2}{2mr_{min}^2} + V(r_{min}) = \frac{1}{2}mv_0^2 \\ l = m r_{min}^2 \dot{\phi} = m\rho v_0. \end{cases}$

for differential cross section, we should have:

$\frac{d\rho}{d\chi}$ what's the real variable during this process?

say: v_0 is fixed yet ρ is not. $\Leftrightarrow \begin{cases} E \text{ fixed} \\ l \text{ not fixed}. \end{cases}$

It is assumed that the 3D electron cloud (that was created in the laser focus) possesses cylindrical symmetry and hence the 3D distribution can be described with only two variables: (p_ρ,p_z). It is a reasonable assumption in case the laser field is linearly polarized (with the axis of the cylinder being the laser polarization axis), and the laser pulse envelope is long enough to contain more than a few optical cycles. A single quadrant of the raw images then contains all the information that can be obtained in the measurement, and hence data from all four quadrants can be added up ("folded" onto one quadrant) to increase signal-to-noise.
The next step is to apply Abel-inversion. Taking advantage of the cylindrical symmetry again, the full 3D distribution can be reconstructed in momentum space from the 2D distribution ($\eta(p_y, p_z) \xrightarrow{Abel-invert} \zeta(p_\rho, p_z)$). The geometry is illustrated in figure 2.20.

$$\zeta(p_\rho, p_z) = -\frac{1}{\pi}\int_{p_\rho}^{\infty} \frac{\partial\eta}{\partial p_y} \frac{dp_y}{\sqrt{p_y^2 - p_\rho^2}} \tag{2.13}$$

Note that the inverse-Abel transformation as given in eq. 2.13 couples the p_x and p_y variables ($p_\rho^2 = p_x^2 + p_y^2$), but not p_z, which then can be treated as a parameter. The transformation should be then performed in planes of (p_x, p_y) for every p_z.

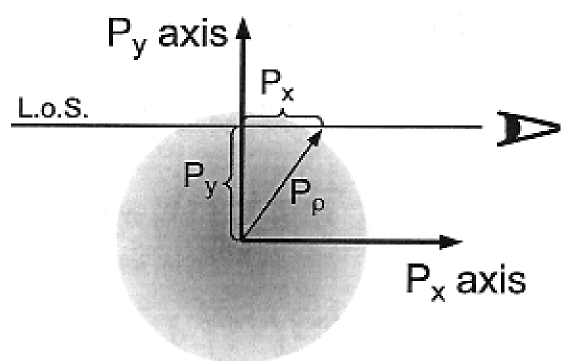


Figure 2.20: A geometrical interpretation of the Abel transform in two dimensions. An observer looks along a line parallel to the p_x-axis a distance p_y above the origin (Line of Sight - L.o.S.). The observer sees the projection (i.e. the integral along p_x) of the circularly symmetric function $\zeta(p_\rho, p_z)$ (for a given value of p_z; function is represented in gray) along the line of sight onto p_y. The observer is assumed to be located infinitely far from the origin so that the limits of integration are $\pm\infty$.

Numerical implementation of the inversion can be done in a number of different ways, all of which have strengths and weaknesses in terms of accuracy, speed and robustness to noise. Apart from straightforward evaluation of eq. 2.13, other algorithms worth noting are the BASEX [27], pBASEX [28] and Onion Peeling [29] methods, of which the BASEX approach was mostly used in chapters 3 and 4 of this thesis.

The total electron yield can then be evaluated from the original distribution in the cylindrical coordinate system:

$$N_{tot} = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_0^{2\pi} \zeta(p_\rho, p_z)p_\rho d\phi\, dp_z\, dp_\rho$$

Given the usually circular nature of the raw detected 2D distributions and the underlying spherical nature of the 3D electron cloud, one can choose to further transform the coordinate system to be spherical:

Final — PHY501, Fall 2016

1. *(40 pts)* A particle of mass m moves on a surface of a cone placed upside down vertically in a gravitational field g. The cone has apex angle 2θ.

(a) Write down the Lagrangian of the particle using as the coordinates the distance r to the cone apex and angle ϕ of rotation around the axis of the cone.

(b) Derive the Lagrange equations of motion for these coordinates.

(c) Find the conserved quantity M associated with the rotational symmetry of the cone for rotations around the axis. State in one sentence what physical variable M represents.

(d) Obtain expression for the energy E of the particle.

(e) For a given M, find the stationary distance r_0 at which the particle can rotate around the cone axis without changing r_0.

2. *(30 pts)* The second-order parametric resonance in a harmonic oscillator of frequency ω_0 is described by the equation of motion for the oscillator coordinate x:

$$\ddot{x} + \omega_0^2 x = -u\omega_0^2 \cos \nu t x\,, \quad \nu = \omega_0 + \epsilon\,, \quad \epsilon \ll \omega_0\,,$$

where $u \ll 1$ is the parameter modulation amplitude and ϵ is detuning of the modulation frequency ν from the frequency ω_0. In the lowest nonvanishing order of the perturbation theory in small modulation amplitude u, find the boundaries $\epsilon_\pm$ of the range of detuning values, $\epsilon \in [\epsilon_-\,, \epsilon_+]$, where the parametric resonance exists at this modulation frequency.

3. *(30 pts)* A particle of mass m and coordinate x is bouncing back-and-forth in a one-dimensional box (infinite square well potential), $x \in [0, L]$. The wall of this box at $x = L$ moves with a small velocity $\dot{L}$, so that its size $L(t)$ changes with time adiabatically, $\dot{L} \ll v$.

(a) If the particle initial velocity is v_0, when the box size is L_0, find the velocity $v(L)$ as a function of L by considering periodic reflections of the particle from the slowly moving wall of the box.

(b) Find $v(L)$ from the adiabatic invariant $I = (1/2\pi)\oint p dx$ for this particle, and compare the result to that in part (a).