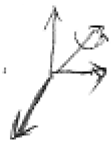
QM 2

I. continuous symmetry

1.  $R_{x,y,z}(\varepsilon) \approx 1 + \varepsilon G_{x,y,z}$.

$G_z = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 0 \end{pmatrix} = -E_{xy} + E_{yx} \Rightarrow G_i = -E_{jk}\varepsilon_{ijk}, \ [G_i, G_j] = \varepsilon_{ijk} G_k$

2. in QM:

① $|\alpha\rangle_R = D(R)|\alpha\rangle \Rightarrow \underset{\sim\sim\sim}{\|\alpha_R\| = \|\alpha\|} \Rightarrow$ we should get a symmetry!

$\Rightarrow D^\dagger D = I \xrightarrow[\text{accepted}]{\text{Taloy is}}$ ~~$D = I + \dots$~~ $D(1+\varepsilon G_z) = I + (-i\varepsilon \frac{J_z}{\hbar})$.

② and notice that such a D forms a group. $\underset{\sim\sim}{SU(2)}$. (why is this 2?)

$SO_3 \xrightarrow{\text{map } D} SU(2)$

③ $[J_i, J_j] = i\hbar\varepsilon_{ijk} J_k$. $\qquad J_\pm = J_x \pm iJ_y$

$[J_z, J_\pm] = \pm\hbar J_\pm$

3. spin - 1/2:

$D(R_{\vec{n}}(\varepsilon)) \approx I - i\varepsilon \frac{\vec{n}\cdot\vec{J}}{\hbar} \underset{\text{spin}\frac{1}{2}}{=} I - i\varepsilon\frac{\vec{n}\cdot\vec{S}}{\hbar} = I - i\varepsilon\frac{\vec{n}}{\hbar}\cdot\frac{\hbar}{2}\vec{\sigma}$

$\Rightarrow D(R_{\vec{n}}(\phi)) = \exp[-i\phi\frac{\vec{n}}{2}\cdot\vec{\sigma}] = \exp[-i(\phi/2)\cdot\sigma_n]$

$= \cos\phi \cdot I - i\sin\phi\, \sigma_n$.

a trick here is $\sigma_a\sigma_b = \delta_{ab} + i\varepsilon_{abc}\sigma_c$

4. spin - precession: $\boxed{\text{what's the relation of this part and Gauge Transformation?}}$ $\boxed{\text{the homework!}}$

$\vec{B}\uparrow$ $\qquad |\psi(0)\rangle = |y+\rangle$.

solution: $H = -\vec{\mu}\cdot\vec{B} \qquad \vec{\mu} = g\cdot\frac{e}{2mc}\cdot\vec{S} \approx \frac{\hbar e}{2mc}\vec{\sigma}$

$= -\frac{\hbar}{2}\omega_c \sigma_z$.

$\Rightarrow U = \exp(-i(H/\hbar)t) = \exp(i\frac{\omega_c t}{2}\sigma_z)$

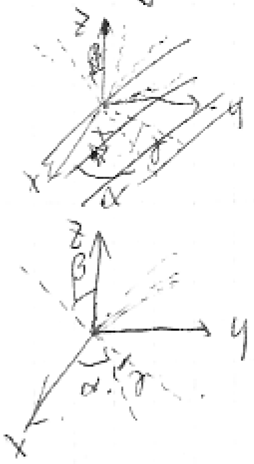
$= \cos(\frac{\omega_c t}{2})\cdot I + i\sin(\frac{\omega_c t}{2})\cdot\sigma_z$.

$\begin{cases} |\psi(t)\rangle = U|\psi(0)\rangle = \cos\omega_c t\,|y+\rangle + i\sin\omega_c t\,|y-\rangle. \\ \langle\vec{S}\rangle = (-\frac{\hbar}{2}\cos\omega_c t, \ \frac{\hbar}{2}\sin\omega_c t, \ 0). \end{cases}$

So it is just very different from CM, where $\vec{\mu}$ is aligned ~~with~~ into the $\vec{B}$'s direction?

2.

5 - Euler Angle.



$R(\alpha,\beta,\gamma) = R_z(\alpha)\cdot R_x(\beta)\, R_z(\gamma)$ ✓

it seems that you could use the math tricks to prove it! n=1 . n=n+1 ...

remember this form!

or $\underbrace{R = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)}$. (we use this).

6 - Angular momentum.

rotation symmetry : $i\hbar\,\partial_t \vec{J} = [\vec{J}, H] = 0$. (H is for free particle).

$[J_i, J_j] = i\hbar\,\varepsilon_{ijk} J_k$ $\qquad \underbrace{SU(2)}$

$[J^2, J_i] = 0.$ $\qquad \vec{J}^2 = \sum J_i^2$

minimal set : $\{J^2, J_z\} \Rightarrow$

$$\begin{cases} J^2 |j,m\rangle = \hbar^2 j(j+1)\,|j,m\rangle & (j = -|j| \text{ is lowed}) \\ J_z |j,m\rangle = \hbar m\,|j,m\rangle & (j(j+1) \geq m^2) \end{cases}$$

$J_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\; |j, m\pm 1\rangle \qquad m \in [-|j|, |j|]$.

$\boxed{\text{question: why the relation here just looks like H.O.?}}$

7 - ~~rotation~~ in jm- basis :

$\langle j,m|\, D(\alpha,\beta,\gamma)\,|j,m'\rangle \triangleq D^j_{mm'}(\alpha,\beta,\gamma)$.

$e^{i(\alpha m - \gamma m')} \cdot \langle j,m|\, e^{-i\frac{J_y}{\hbar}\beta}\,|j,m'\rangle \triangleq e^{i(\alpha m - \gamma m')}\, d^j_{mm'}(\beta)$

$$\Rightarrow D = \begin{bmatrix} [D(1/2)] & & \\ & [D(1)] & \\ & & [D(3/2)] \end{bmatrix}$$ semi - diagnalized

8 - orbital angular momentum :

① spinless : $\vec{J} = \vec{L}$ $\qquad \vec{L} = \vec{x}\times\vec{p} = \varepsilon_{ijk} x_i p_j$.

spin : $\vec{J} = \vec{L} + \vec{S}$ $\qquad \downarrow$

$[L_i, L_j] = i\varepsilon_{ijk}\hbar L_k$ $\quad \underbrace{SU(2)}$.

question: why is the "L" obeys SU(2). which is for all dim, I mean, for n-dim, we could have n-1 symmetry, then where is till the other ~~symmetry~~ symmetries?

then we get the expression for $\vec{L}$, L_z and L^2:

$$\vec{L} = -i\hbar(\hat{\phi}\,\partial_\theta - \frac{\hat{\theta}}{\sin\theta}\partial_\phi). \qquad L_z = -i\hbar\partial_\phi$$

$$L^2 = \cancel{-\hbar^2(\ldots)} \; -\hbar^2\left(\frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta)\right)$$

and back to geometry (physics):

$$H = \frac{P^2}{2m} = \frac{P_r^2}{2m} + \frac{L^2}{2mr^2}$$

fix r, the eigenvalue/vector give you the angular distribution

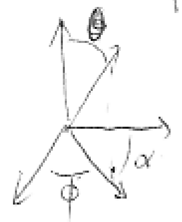
it is more convienent to do this on the surface $r=1$.

i.e. $\int d\Omega \cdot |e^{im\phi} P_l^m(\cos\theta)|^2 = 1$.

this defines the overall "amplitude" of the angular distribution.

② now turn to the relation to D. matrix:

$$\varphi_l^m(\hat{r}) = \langle lm|\hat{r}\rangle = \langle lm|D(\tfrac{\pi}{2}-\phi, \theta, 0)|\hat{z}\rangle$$



$$= D^l_{mm'}(\tfrac{\pi}{2}-\phi, \theta, 0)\;\langle lm'|\hat{z}\rangle$$

$$= D^l_{mm'}(\tfrac{\pi}{2}-\phi, \theta, 0)\;\delta_{m'0}\cdot\sqrt{\frac{2l+1}{4\pi}}\cdot\frac{1}{2}\cdot P_l(1)$$

9 — addition theorem:

conservation of total momentum

$[H, J = \vec{J}_1 + \vec{J}_2] = 0 \implies$ rotational symmetry conserved. $\vec{r}\cdot\vec{J}$ would be violent

$\implies$ minimal set $\{J^2, J_z, J_1^2, J_2^2\}$ $\quad |J\,m\,J_1 J_2\rangle$

another minimal set $\{J_1^2, J_{1z}, J_2^2, J_{2z}\}$ $\quad |J_1 m_1\rangle \otimes |J_2 m_2\rangle$ — Unitary

$$|J m J_1 J_2\rangle = \sum |J_1 m_1 J_2 m_2\rangle \underbrace{\langle J_1 m_1 J_2 m_2|J m J_1 J_2\rangle} \rightarrow \text{CG factor}$$

10 — Tensor:

$$H = \alpha V_i V_i + \beta V_{ij} V_{ji} + \cdots$$

that preserves the rotation symmetry, i.e. $[H, \vec{J}] = 0$

and the irreducible terms in tensor expression would be invariant.

~~2017/2/2~~. 2/2/2017

① rank-2:

$$D_{ij} = \frac{1}{3}\delta_{ij}(T_{mm}) \qquad 1\#$$

$$+\frac{1}{2}(T_{ij} + T_{ji} - \frac{2}{3}\delta_{ij}(T_{mm})) \qquad 3\#$$

$$+\frac{1}{2}(T_{ij} - T_{ji}). \qquad 5\#$$

$T_{mm} = \sum T_{ii} = \mathrm{tr}\,T$

these components are all tensors, and irreducible. i.e. $T_{ij} = C_K^k T^k_{(ij)}$

②. cartesian coordinate $\longleftrightarrow$ spherical coordinate.

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$(e_x,\ e_y,\ e_z) \xleftrightarrow{?} (e_+,\ e_0 = e_z,\ e_-).$$

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad \begin{cases} R_z(\phi)\, e_z = e_z. \\ R_z\, e_\pm = e^{\mp i\phi}\, e_\pm . \end{cases}$$

11- tensors in QM.

· Integer spin + irreducible + spherical coordinate = transform like ~~K_m^l~~ Y_l^m.

$$(Y^m_l)^R = \langle x | l m \rangle_R = D^{l\,*}_{mm'}\, Y^{m'}_l(x).$$

T^k_q = irreducible representation of rank k. in spherical representation.
with $q = -k, \cdots k$.

$$(T^k_q)^R = D^{k\,*}_{qq'} T^k_{q'}$$

essense of this relation:

☆ $$(T^k_q)^R = D^\dagger(R)\, T^k_q\, D(R) = D^*_{qq'}\, T^k_{q'}$$

and in infinitesimal frame:

$$D(R) = 1 - i\varepsilon J_\varepsilon/\hbar \Rightarrow (1 + iJ_\varepsilon/\hbar)\, T^k_q\, (1 - iJ_\varepsilon/\hbar) = \langle kq' | 1 + iJ_\varepsilon/\hbar | kq \rangle\, T^k_{q'}$$

$$\Rightarrow [J_\varepsilon, T^k_q] = \langle kq' | J_\varepsilon | kq \rangle\, T^k_{q'}.$$

$$\Rightarrow \begin{cases} [J_z, T^k_q] = \hbar q \cdot T^k_q. \\ [J_\pm, T^k_q] = \hbar\sqrt{k(k+1) - q(q\pm1)}\; T^k_{q\pm1}. \end{cases}$$

? $\boxed{\text{this seems to be connected to the eigenstates and operators ?}}$

12- a theorem on two sets of irreducible representation.

we have: $X^{k_1}_{q_1}$, $X^{k_2}_{q_2}$, then we can construct: ~~cleb~~ clebsch

$$X^k_q = \sum_{q_1 q_2} \langle k_1 k_2 q_1 q_2 | \cancel{k_1 k_2 q_1 q_2}\; k_1 k_2 k q \rangle\, X^{k_1}_{q_1} X^{k_2}_{q_2}. \quad (q_1 + q_2 = q)$$

? proof: $D^\dagger X^k_q D \stackrel{\triangle}{=} \sum \langle k_1 k_2 q_1 q_2 | k_1 k_2 k q \rangle\, D^{k_1\,*}_{q_1 q_1'} D^{k_2\,*}_{q_2 q_2'}\, X^{k_1}_{q_1'} X^{k_2}_{q_2'}$

use the identity: $D^{k_1}_{q_1 q_1'} D^{k_2}_{q_2 q_2'} = \sum \langle k_1 k_2 j m | k_1 k_2 q_1 q_2 \rangle \langle k_1 k_2 j m' | k_1 k_2 q_1' q_2' \rangle\, D^j_{mm'}$

$$\Rightarrow D^\dagger X^k D = D^{k\,*}_{qm'}\, X^k_{m'}$$

☆ 13 - Wigner - Eckart theorem. (also look at ~~hydrogen atom~~ P.7 part),

$$\langle \alpha' j' m' | T^k_q | \alpha j m \rangle = \underbrace{\frac{\langle j k m q | j k j' m' \rangle}{\sqrt{2j+1}}}_{\text{kinematic}} \underbrace{\langle \alpha' j' \| T^k \| \alpha j \rangle}_{\text{dynamic}}$$ azimuthal ?

shell.

kinematic ⇓ gives you the selection rules !!!

~~proof~~: proof:

$$D^\dagger T^k_q D = D^k_{qq'}{}^* T_{q'}$$

write its infinitesmal structure (Lie structure). then:

$$[\vec{J}\cdot\vec{\varepsilon}, T^k_q] = \langle kq' | J_\varepsilon | kq \rangle T^k_{q'}$$

especially:

$$[J_\pm, T^k_q] = \hbar\sqrt{k(k+1) - q(q\pm1)}\, T^k_{q\pm1}$$
$$\triangleq \hbar\, C^\pm_{kq} T^k_{q\pm1}.$$

$$T_z^1 = \frac{\langle S_z \rangle}{\langle J_z \rangle} = \frac{\langle S_n \rangle}{\langle J_n \rangle} = T_n^1$$
$$\langle S_z \rangle = T_z^1, \quad \langle S_n \rangle = T_n^1$$

q just looks like some kind of angular momentum. $\vec{S}$ so is k, thus we could imagine k and q to be some converted j and S in some way.

then ~~$\langle \alpha' j' m' | T^k_q$~~

$$\langle \alpha' j' m' | [J_\pm, T^k_q] | \alpha, j, m \rangle = \hbar C^\pm_{kq} \langle \alpha' j' m' | T^k_{q\pm1} | \alpha j m \rangle$$
$$= C^\mp_{j'm'} \langle \alpha' j' m' \mp 1 | T^k_q | \alpha j m \rangle - C^\pm_{jm} \langle \alpha' j' m' | T^k_q | \alpha j m\pm1 \rangle$$

Same identity:

$$C^\mp_{j'm'} \langle jkmq | jk\, j' m' \mp 1 \rangle - C^\pm_{jm} \langle jk\, m\pm1\, q | jk j' m' \rangle$$
$$= C^\pm_{kq} \langle jkm\, q\pm1 | jk j' m' \rangle. \quad (j + k \Rightarrow j')$$

$(\langle \cdot\cdot | T | \cdot\cdot \rangle \Leftrightarrow \langle jkmq | jkj'm' \rangle.)$ (refers to ~~the~~ j recoupling).

II. discrete symmetries.

1. general way is: $\frac{\partial \mathcal{L}}{\partial \dot{q}} = \vec{p}$ (continuous) then ~~at~~ transform is.

$$R = e^{-i \frac{\vec{q}\cdot\vec{p}}{\hbar}}$$ yet not work here! for discrete!

2. parity. i.e

$$\vec{X} \to M\vec{X} = -\vec{X}, \quad \text{i.e. } M = -I$$

yet in Hilbert space,

$$\langle \vec{x} | (\Pi | \varphi \rangle) = \langle x | \varphi' \rangle.$$ is not so simple.

and we see ~~that~~ $\Pi^2 = I \Rightarrow \Pi^{-1} = \Pi$.

6.

~~and also~~ and also:

$$\Pi^{\dagger}\Pi = I. \qquad (\| \Pi|x\rangle \| = \| |x\rangle \|).$$

finally we arrive: $\psi(x') = \psi(-x)$.

2 if the $[H, \Pi] = 0 \Rightarrow \Pi$ is a symmetry, share the same eigen

$$\begin{cases} \Pi|\psi_\pm\rangle = \pm|\psi_\pm\rangle \\ H|\psi_\pm\rangle = E_\pm|\psi_\pm\rangle \end{cases}$$

3- examples.

① $V = \begin{cases} 0, & |x| \le a \\ \infty & |x| > a. \end{cases}$ $\qquad H = H_0 + \frac{p^2}{2m}$ $\qquad \cancel{[\Pi, X] = -X}$, $\cancel{[P, \Pi] = +P}$?

question: what's the commutator of Π and other? like P. J. ?

② $H = L^2/2I$. $\qquad \Pi L_i \Pi = \Pi^{\dagger} L_i \Pi = L_i$ $\qquad$ i.e. $[L_i, \Pi] = 0$.

$$\langle l m | \Pi | x \rangle = Y^l_m(-x) = Y^l_m(\pi - \theta, \pi + \varphi).$$
$$= (-1)^m e^{im\phi} P^m_l(\theta) \cdot (-1)^{l+m} = (-1)^l Y^l_m(\theta, \varphi).$$

4- operators.

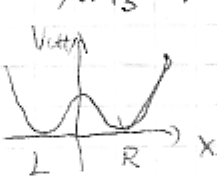
$\{\Pi, X\} = \cancel{X}\, 0$ $\qquad \{\Pi, P\} = -\,0.$

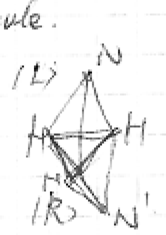
$[\Pi, L_i] = 0$ $\qquad$ $\cancel{[\Pi, \vec{x}\cdot\vec{S}] = 0}$ $\{\Pi, \vec{X}\cdot\vec{S}\} = 0$

$\cancel{[\Pi, \vec{L}\cdot\vec{S}] = }$ $\qquad$ $[\Pi, P^2] = 0.$

$[L^2, \Pi] = 0.$ $\qquad$ $\Pi\, \vec{L}\cdot\vec{S}\, \Pi = -\vec{L}\cdot\vec{S}.$

5. - NH_3 molecule.





take N as a variable.

effective Hamiltonian. 2x2 $H = \begin{pmatrix} \varepsilon & -\Delta \\ -\Delta & \varepsilon \end{pmatrix}$

$E_\pm = \varepsilon \pm \Delta \qquad |\pm\rangle = \frac{1}{\sqrt{2}}(|L\rangle \mp |R\rangle).$

$\Pi|\pm\rangle = \pm|\pm\rangle$ $\qquad$ and $\Pi V_{eff} \Pi = V_{eff}$.

and let $|\psi(0)\rangle = |L\rangle$.

$\Rightarrow U|\psi(0)\rangle = |\psi(t)\rangle = \cancel{\cos(2\Delta t/\hbar)|L\rangle + i\sin(2\frac{\Delta t}{\hbar})|R\rangle}$

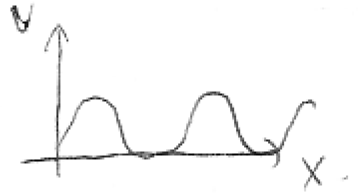
$$= e^{-i\Delta t/\hbar} \cdot [\cos(2\Delta t/\hbar)|L\rangle + i\sin(2\Delta t/\hbar)|R\rangle]$$

02/14/2017

6 - discrete translation → crystal symmetry.

$T_{\vec{a}}$ where $\vec{a}$ is discrete to give you $[T_{\vec{a}}, H] = 0$.

i.e.



Bloch theorem:

$$\begin{cases} \psi(x) = e^{ikx} U_k(x) \\ U_k(x+a) = U_k(x) \\ \psi(x+a) = e^{ika} \psi(x) \end{cases}$$

proof is based on: $[T_a, H] = 0$.

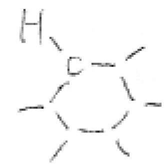
$$\Rightarrow \begin{cases} H|\psi\rangle = E|\psi\rangle \\ T_a|\psi\rangle = \lambda_a|\psi\rangle \end{cases}$$ and defination $T_a|x\rangle = |x+a\rangle$

and if ① periodic. $\lambda_a^N = 1 \Rightarrow \lambda_a = e^{i\frac{2\pi p}{N}}$

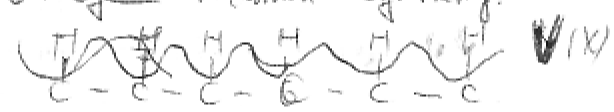
then $\psi(x+a) = \cdots \psi(x)$

Then ② make this $N \to \infty$. all green. $k = \frac{2\pi p}{Na}$

7 - C_6H_6 Benzene (electron's energy).



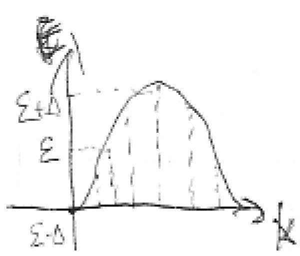
N = 6. ~~symme~~ rotational symmetry.



H~~ükel~~ this is for tight-bounding

remark: you can understand this by knowing there as a bunch of 6 atoms. In some case we degenerate some freedom, just left the "symmetric motion". thus we could write down:

$$H = \varepsilon I - \Delta \sum (E_{i(i+1)} + E_{i(i-1)})$$



where ε is normal potential, yet Δ is for tunneling.

and let $|\psi_k\rangle = \lambda_{nk}|n\rangle$

$$\Rightarrow \lambda_{(n+1)k} = \lambda_{nk} \cdot e^{-i\frac{2\pi}{N}} \triangleq \lambda' \quad \checkmark$$

$$E(k) = \varepsilon - 2\Delta\cos(ka). \quad k = \frac{2\pi p}{Na} \quad \checkmark$$

remark to the tenfold sth: main/basic idea is the generic Hamiltonian and its related symmetry, based on some model, we could try to find the most foundamental Hamiltonians. but right now when we are thinking of superconductivity, we could have more. Randomness and Gaussian distribution.

eigenstates: $|\psi\rangle = \sum \lambda_i^n |n\rangle \qquad \lambda_i = (\lambda)^i$

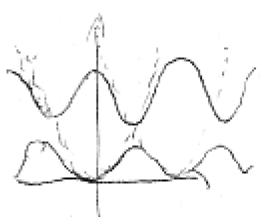
and dispersion relation tells you the influence of the symmetry to the quantization (eigenstates splits, change of basis !!!)

Just imagine $Na = \beta$ const yet $N \to \infty$, we have a "continuous" spectrum.

8

$E(k) \triangleq \varepsilon - 2\Delta\,(1 - \frac{1}{2}(ka)^2)$.

$= \varepsilon - 2\Delta + \frac{(ka)^2}{2(\frac{1}{2\Delta})}$ → look like momentum. → kinetic mass !!!



B-zone

and this expression tells us why we could use free particle's Hamiltonian to calculate the ~~relation~~ SM of phonons.
and this shows our modification to the potential structure !!!
everything is clear after that !!!

$\boxed{\begin{array}{l}\text{question : what if our Hamiltonian looks like :} \to \\ H = \varepsilon I - \Delta_1 E_{i(i\pm)} - \Delta_2 E_{i/i\pm 2)} - \Delta_3 E_{i(i\pm 3)} - \cdots \\ \text{we could expect what kind of output?}\end{array}}$

8 — time-reversal ~~symm~~ symmetry. (V is t-independent)

ass0. our basic equation for motion is Newton / Schrödinger equation

$\Rightarrow \quad i\hbar\partial_t\psi = H\psi$. $\boxed{\text{why not use this expression but * ?}}$

ass1. $\Rightarrow \quad -i\hbar\partial_t\psi^* = H\psi^* = i\hbar\partial_{t'}\psi^*(-t;x) = H\psi^*(t',x)$.

as a summary : $\begin{cases}\psi \xrightarrow{I} \psi^*(-t,x) \\ H \xrightarrow{I} H\end{cases} \quad \to [\Theta ; H] = 0$

In fact these are anti-unitary symmetry

9 — . Wigner theorem : A symmetry is measurement preserving transform.

to be precise : $|\langle\hat\alpha|\hat\beta\rangle| = |\langle\alpha|\beta\rangle|$.

$\Rightarrow \langle\hat\alpha|\hat\beta\rangle = \begin{cases}\langle\alpha|\beta\rangle & \text{unitary} ✓ \\ \langle\beta|\alpha\rangle. & \text{anti-unitary} ✓\end{cases}$

☆. transform to operator : $A \to \Theta A^{+}\Theta^{-1} = \hat{A} = \Theta(A)$ (think of Heisenberg picture).

$\begin{cases} X \to \Theta X^{+}\Theta^{-1} = X \\ P \to \Theta P^{+}\Theta^{-1} = -P \\ \vec{L} \to \Theta \vec{L}^{+}\Theta^{-1} = -\vec{L} \\ H \to \Theta H^{+}\Theta^{-1} = H.\end{cases}$

$\Theta|x\rangle = |x\rangle$

$\Theta|p\rangle = |-p\rangle$

$\Theta|j,m\rangle = |j,-m\rangle(-1)^m$

10 — (1) specific representation for Θ : $S = 1/2$.

!!!

$\Theta = \xi e^{-i\pi S_y/\hbar}\vec{K}$ (conjugate ~~p~~ operator).

$= -i\xi\sigma_y\vec{K}$ (ξ is a $|\xi|^2 = 1$ constant).

$\Rightarrow \Theta^2 = -I$.

$\Theta S_i \Theta^{-1} = -S_i$

(2). for any J, Θ look like

$$\Theta = \xi e^{-i\pi J_y/\hbar} \vec{K}$$

$$\Theta^2 = |\xi|^2 e^{-i\pi(J_y - J_y^*)/\hbar} \vec{K}^2 = e^{-2i\pi J_y/\hbar}$$

make compare $\Theta|l,m\rangle = (-1)^m |l,-m\rangle$

$$\Theta^2 |l,m\rangle = (-1)^{2m} |l,m\rangle = \begin{cases} |l,m\rangle & \text{integer} \\ -|l,m\rangle & \text{anti-integer} \end{cases}$$

remark: Thomas process would give you this, i.e. spin ↑ would run a circle and finally give you a spin ↓ (not spin down, but orientation ↓).

III. 3-dim problem

1 - Spherical Symmetry $[D(R), H] = 0$.

$H = \frac{p^2}{2m} + V$. then since $[D(R), \frac{p^2}{2m}] = 0$.

$\Rightarrow [V, D(R)] = 0$.

$\Rightarrow V(R^{-1}\vec{x}) = V(\vec{x})$. $\Rightarrow V = V(|\vec{x}|)$ (e.g. $V = -\frac{e^2}{|\vec{x}|}$)

to be more precise:

$H = \frac{p^2}{2m} + V$ $\quad \vec{P} = P_r \hat{r} + P_\perp \hat{n}_\perp$

$$\begin{cases} \vec{P_r} = \hat{r} \cdot \frac{1}{2}(\hat{r}\cdot\vec{p} + \vec{p}\cdot\hat{r}) & \text{(order matters!)} \\ \vec{P_\perp} = (\hat{r}\times\vec{p})\times\hat{r} & \text{(no matter how to you choose order)} \end{cases}$$

$\Rightarrow \frac{p^2}{2m} = \frac{P_r^2}{2m} + \frac{L^2}{2mr^2}$

$\Rightarrow H|E_n l m\rangle = E_n |E_n l m\rangle$

$L^2 |E_n l m\rangle = \hbar^2 l(l+1) |E_n l m\rangle$ $\quad$ power of symmetry!!

$L_z |E_n l m\rangle = \hbar m |E_n l m\rangle$

$\Rightarrow H = \frac{P_r^2}{2m} + \frac{\hbar^2}{2mr^2} l(l+1) + V(r)$.

we could replace L^2 by $\hbar^2 l(l+1)$ because $[L^2, H] = 0$ i.e. it doesn't influence other parts of the Hamiltonian.

2 - In D-dim, we see: P_r.

$$P_r = \frac{1}{2}(\vec{p}\cdot\hat{r} + \hat{r}\cdot\vec{p}) = \frac{\hbar}{i}\left(\partial_r - \frac{D-1}{2r}\right)$$

☆ $\Rightarrow P_r^2 = -\hbar^2\left(\partial_r^2 + \frac{D-1}{r}\partial_r + \frac{(D-1)(D-3)}{4r^2}\right)$

10.

3 — Hydrogen

$[H = -\frac{\hbar^2}{2m}(\partial_r^2 + \frac{2}{r}\partial_r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} - \frac{e^2}{r}] | E \ell m\rangle = E | E \ell m\rangle$

$\langle x | E \ell m\rangle = R_n(r) \cdot Y_\ell^m(\hat{x})$.

let $-\frac{1}{r_0^2} \triangleq E \cdot 2m/\hbar^2$ $\quad \rho \triangleq r/r_0$. $\quad \frac{e^2}{r_0 E} \triangleq K$

$\begin{cases} \rho \to 0 & \text{throw away } 1/\rho. \text{ maintain } \partial_\rho^2, \frac{1}{\rho}\partial_\rho, 1/\rho^2. \\ \rho \to \infty & \text{throw away all other three, maintain } \partial_\rho^2. \end{cases}$

$\left.\begin{matrix} \rho \to 0 & \rho^\ell \\ \rho \to \infty & e^{-\rho} \end{matrix}\right\} \Rightarrow R \propto \rho^\ell e^{-\rho} \varphi(\rho)$.

question: 1. see it in propagator?
2. if we have an ladder ~~op~~ operator for this R?

$\Rightarrow$ Coulumb equation:

$-\rho\varphi'' - 2[(\ell+1) - \rho]\varphi' - (K - 2(\ell+1))\varphi = 0$

$\Rightarrow$ taylor expansion, we finally have:

$a_{n+1}/a_n = -(K - 2(\ell+n+1)) / (\ell(\ell+1) + 2(\ell+1)(\ell+1))$

$\Rightarrow K = 2(\ell+n+1)$ for some n, ℓ. then let $N = n+\ell+1$ ✓

we could solve r_0, E, from def of K and r_0

$\Rightarrow E_N = -\frac{me^4}{2\hbar^2} / N^2 \triangleq -R_y \cdot 1/N^2$

notice we have an important factor here:

$R_y = \frac{me^4}{2\hbar^2} = \frac{1}{2}mc^2(\frac{e^2}{\hbar c})^2 \triangleq \frac{1}{2}mc^2\alpha^2$ (Fine structure constant)

4 — Bohr.

most important assumption: $L = n\hbar$

Virial:

$E = \frac{p^2}{2m} + V$ $\quad$ let $x \to \lambda x$.

$E \to E(\lambda) = \frac{K}{\lambda^2} + \lambda^\alpha V$.

$\frac{dE}{d\lambda} = -2\frac{K}{\lambda^3} + \alpha\lambda^{\alpha-1}V = 0$. $\Rightarrow K = \frac{\alpha}{2}V$ ...

remark: this is in fact telling about transformation, i.e. symmetry.

$\Rightarrow E = -\frac{R_y}{n^2}$

IV time-independent perturbation theory

I. our problem, Hamiltonian

$H = H_0 + V \longrightarrow$ we don't quite know, but time independent

$H_0|n\rangle = E_n|n\rangle$

our idea is to organize the general stationary state based on the eigen states of H_0 and with power of V.

2 — 2X2 system. exact solution.

$$H_{2\times2} = \begin{pmatrix} \varepsilon_1 & V \\ V & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \\ & \varepsilon_2 \end{pmatrix} + V \cdot \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \triangleq H_0 + V.$$

$\boxed{\text{question : how to organize our problem in order to get } tr(H_0) = 0 \text{ ? i.e we set } |\psi\rangle = |\psi_0\rangle + |\psi_1\rangle \text{ then } \Leftrightarrow H|\psi_1\rangle = E|\psi_1\rangle \text{ where } tr(H_0) = 0.}$

ass1 $\quad \lambda_\pm \overset{V}{\underset{\text{small}}{=}} \dfrac{\varepsilon_1+\varepsilon_1}{2} \pm \frac{1}{2}\sqrt{(\varepsilon_1+\varepsilon_2)^2 - 4(\varepsilon_1\varepsilon_2 - V^2)}$

$$= \begin{cases} \varepsilon \pm |V| & \varepsilon_1 = \varepsilon_2 \\ \dfrac{\varepsilon_1+\varepsilon_1}{2} \pm \dfrac{\varepsilon_1-\varepsilon_2}{2} \pm \dfrac{V^2}{\varepsilon_1-\varepsilon_2} \quad \begin{cases} \varepsilon_1 + \dfrac{V^2}{\varepsilon_1-\varepsilon_2} \\ \varepsilon_2 - \dfrac{V^2}{\varepsilon_1-\varepsilon_2} \end{cases} & \varepsilon_1 \neq \varepsilon_2 \end{cases}$$

$\boxed{\text{question : what is a damping in 2X2 system?}}$

3 — ~~Raile~~ Rayley — schrödinger equation.

the expression we are seeking ~~$\langle V \rangle$~~ ~~$\langle n|V|n\rangle = 0$~~

$$(H_0+V)(|\overset{0}{n}\rangle + |\overset{1}{n}\rangle + \cdots) = (\overset{0}{E_n} + \overset{1}{E_n} + \cdots)(|\overset{0}{n}\rangle + \cdots)$$

① V^0 : $H_0|\overset{0}{n}\rangle = \overset{0}{E_n}|\overset{0}{n}\rangle \Rightarrow \overset{0}{E_n} = E_n$

② V^1 : $H_0|\overset{1}{n}\rangle + V|\overset{0}{n}\rangle = \overset{0}{E_n}|\overset{1}{n}\rangle + \overset{1}{E_n}|\overset{0}{n}\rangle \Rightarrow E_n^1 = \langle\overset{0}{n}|V|\overset{0}{n}\rangle$

☆ $\quad \chi_n + |\overset{0}{n}\rangle\langle\overset{0}{n}| \triangleq 1 \quad \Rightarrow |\overset{1}{n}\rangle = \dfrac{\chi_n}{(E_n^0 - H_0)} V|\overset{0}{n}\rangle$ (first order)

$$\Rightarrow |\overset{1}{n}\rangle = \sum \frac{\langle k|\chi_n V|\overset{0}{n}\rangle}{E_n^0 - E_k^0}|k\rangle$$

③ V^2 : $H_0|\overset{2}{n}\rangle + V|\overset{1}{n}\rangle = E_n^0|\overset{2}{n}\rangle + E_n^1|\overset{1}{n}\rangle + E_n^2|\overset{0}{n}\rangle$

$$E_n^2 = \langle\overset{0}{n}|V|\overset{1}{n}\rangle = \sum \frac{|\langle\overset{0}{n}|V|k\rangle|^2}{E_n^0 - E_k^0}$$

$$(E_n^0 - H_0)|\overset{2}{n}\rangle = (V - E_n^1)|\overset{1}{n}\rangle - E_n^2|\overset{0}{n}\rangle$$
$$= (\chi_n V - E_n^1)|\overset{1}{n}\rangle$$
$$\Rightarrow |\overset{2}{n}\rangle = \frac{1}{E_n^0 - H_0}(\chi_n V - E_n^1)|\overset{1}{n}\rangle$$

4 — degenerate perturbation theory

①. E_n has degeneracy $|n_1\rangle \cdots |n_\Delta\rangle$

then we ~~diagonalize~~ diagonalize V in ~~this~~ Δ subspace to lift degeneracy

$V = \begin{pmatrix} \ddots \end{pmatrix}$. $C^+ V C = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_\Delta \end{pmatrix}$ change $|n_i\rangle$ to $|\tilde{n}_j\rangle = C_{ij}|n_i\rangle$

thus they are no longer degenerate anymore.

12.

② repeat the perturbation theory for non-degenerate cases.

5 – Quadratic Stark Effect.

$\Delta E = -\vec{p}\cdot\vec{E}$. $H = \frac{p^2}{2m} - \frac{e^2}{r} - eEz = H_0 - eEz$

use P. T. E_3 ——

E_2 —— $N = 2 = n + \ell + 1$. Linear Stark.

E_1 —— $N = 1 = n + \ell + 1$. $\Rightarrow n = \ell = 0$. quadratic Stark

ground state:

① $\overset{0}{E}_0 = -\frac{e^2}{2a}\cdot\frac{1}{1^2}$

② $\overset{1}{E}_0 = \langle 0|V|0\rangle = \langle 0|-eEz|0\rangle = -eE\langle 0|\Pi z \Pi|0\rangle \Rightarrow 0$

trick ③ $\overset{2}{E}_0 = \langle 0|V\frac{\mathcal{Q}_0}{E_0 - H_0}V|0\rangle = \sum_{k\neq 0}\frac{|\langle 0|V|k\rangle|^2}{E_0 - E_k} = -\frac{1}{2}\alpha_E E^2$

here α_E is electrical polarizability.

$\alpha_E \sim e^2\cdot\frac{z^2}{e^2/a} \sim a^3$

also make use of. $[H_0, z^3] = -\frac{\hbar^2}{2m}\cdot 6z$.

$\Rightarrow \alpha_E = \frac{2me^2}{3\hbar^2}\langle 0|z^4|0\rangle$

where $\langle x|0\rangle = C\cdot e^{-r/a}$. $Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$

trick make use of $\int r^6 e^{-2r/a}\,dr = (\frac{a}{2})^7\cdot 6!$

$\Rightarrow \langle 0|z^4|0\rangle = (\frac{2}{a})^3\cdot\frac{1}{2!}(\frac{a}{2})^7\cdot 6!\cdot\frac{1}{5} = \frac{9}{2}a^4$

$\Rightarrow \overset{2}{E}_0 = -\frac{me^2}{3\hbar^2}\cdot\frac{9}{2}a^4\cdot E^2$

? $= \frac{V^2}{-\delta E} \approx \frac{(eEa)^2}{-e^2/a} = -a^3E$.

6 – Linear Stark Shift.

$N = 2$ $\ell = 0$ $\Rightarrow$ 1 state

$E\hat{z}$ $\Uparrow$ $N = 2$ $\ell = 1$ $\Rightarrow$ 3 states

notation: spectroscopic notation

e.g. $n = 2$ $L = \cdots$ $J = \cdots$

$_nL_J$. $2P_{1/2}$ $2P_{3/2}$ $2S_{1/2}$

$L = 0.\ 1.\ 2.\ 3 \cdots$

S P L D $\cdots$

throw away the spin

· since $\overset{n=2}{\cancel{2P}}$ 4 states are degenerate, we should first do sth. to them. i.e.

$$\langle n\ell' m' | V | n\ell m\rangle = \langle n\ell' m' | -eEz | n\ell m\rangle$$

$$\xrightarrow[\cancel{Sym}\ \text{Tensor}]{\text{parity or}} (-1)^{\ell+\ell'+1} \langle n\ell' m' | -eEz | n\ell m\rangle$$

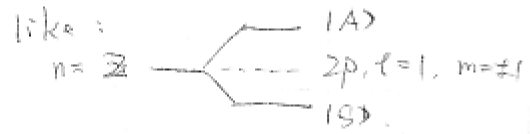
$\Longrightarrow$ $\ell+\ell'=1$ is a must.

and also $m=m'$, in summary. ($z = Y_\ell^m = Y_1^0$).

$$V = \begin{bmatrix} 0 & A & 0 \\ A & 0 & \\ 0 & & 0 \end{bmatrix} \qquad A = \langle 210 | -eEz | 200\rangle$$

Changed eigenstates are $|S(A)\rangle = \frac{1}{\sqrt{2}}(|210\rangle \pm |200\rangle)$

like :



7 — spin – orbit effect.

change frame of E-M field $\Rightarrow$ orbit has both E and B.

i.e.

ass 2. $\begin{cases} \vec{E} = \vec{E}_p' + 0 \\ \vec{B} = 0 + \vec{\beta}\times\vec{E}_p' \end{cases}$ $\vec{E}_p' = -\frac{e}{r^2}\hat{r} = -\nabla |e|\varphi$ (non-relativistic) (created by proton).

ass 3. $V_{int} = -\vec{\mu}_e \cdot \vec{B}$ $\left(\vec{\mu}_e = \frac{g_e e}{2mc}\vec{S}\right)$.

$$= -\frac{e^2}{(mc)^2}\cdot\frac{1}{r}\partial_r \frac{1}{r}\ \vec{L}\cdot\vec{S} \quad \left(= \frac{e^2}{m^2c^2}\frac{\varphi'}{r}\ \vec{S}\cdot\vec{L}\right)$$

estimation : $V_{int} = \left(\frac{e}{mc}\right)^2\cdot\frac{\hbar^2}{a^3} = \alpha^2\cdot\left(\frac{e^2}{a}\right) = \alpha^2\cdot V_{Coulumb}$

this is the reason why it is called hyperfine structure.

> and symmetry tells us that V_{int} is "specialized" for T and J. thus intuitively, n itself doesn't split, yet J. would be influenced. i.e. with the same L, S. the degeneracy would be destroyed.
>
> Important fact: $[\vec{L}, \vec{S}] = 0$. no matter $[J_i, J_j] = i\epsilon_{ijk}J_k$ balabala
>
> this notation is stupid because the real case is $\vec{J} = \vec{L}\otimes I + I\otimes\vec{S}$

perturbation theory : $E_n^1 = \langle njm | \frac{(J^2 - L^2 - S^2)}{2} | njm\rangle$

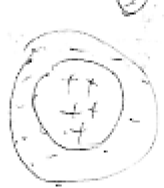
$\Longrightarrow$ $E_n^1(+) = \alpha_n \cdot \frac{\hbar^2}{2}\ell$. $j=\ell+1/2$

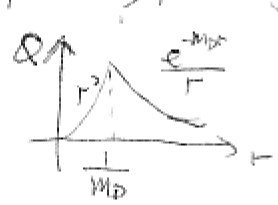
$E_n^1(-) = -\alpha_n \frac{\hbar^2}{2}(\ell+1)$ $j=\ell-1/2$ (for $\ell>0$, for $\ell=0$, no lift up).

$\boxed{\alpha_n \text{ shows sth. in W-E theorem.}}$

· discussion : ① $1/r$ is ~~the~~ accidental degeneracy.

② $1/r^n$, $e^{-m_D r}/r$ charge screening effect., no accidental degeneracy





Debye shielding

14.

8. Zeeman - effect.

ass 1. $\vec{A} = (-\frac{By}{2}, +\frac{Bx}{2}, 0)$. $\vec{B} = B\hat{z}$.

$$H = \frac{(\vec{p} - \vec{A}\frac{e}{c})^2}{2m} + V_{coulomb} + V_{s.o.}$$

question: what about gauge transformation?

$$= \frac{p^2}{2m} - \frac{e}{mc}\vec{A}\cdot\vec{p} + \frac{e^2}{2mc}\vec{A}^2 + V_{coulomb} + V_{s.o.} \quad (\text{use } \vec{p}\cdot\vec{A} = 0.\ \nabla\cdot\vec{A} = 0)$$

$$= \frac{p^2}{2m} - \frac{eB}{mc}L_z + \frac{e^2}{8mc}B^2(x^2+y^2) + V_{coulomb} + V_{s.o.} \quad (\text{use } \vec{A}\text{'s expression}).$$

$$\xrightarrow{\text{spin}-\vec{B}} \frac{p^2}{2m} - \frac{eB}{2mc}(g_L L_z + g_s S_z) + \frac{e^2}{8mc}B^2(x^2+y^2) + V_{coulomb} + V_{s.o.}$$

now let's discuss how to separate our discussion into the spectrum.

i.e. in j base or m ℓ base !?

case 1. B is small.

$$\frac{eB\hbar}{mc} \approx \frac{e^2}{a}\frac{\phi}{\Phi_0} = \frac{e^2}{a}\cdot\frac{Ba^2}{\hbar c/e} << \frac{e^2}{a} \quad (\text{remember Landau problem}).$$

also compare this with $V_{s.o.}$!!!

then the question is: if $V_{s.o.} >> \frac{e^2}{a}\phi/\Phi_0$. use perturbation, we have:

W-E theorem

$$\frac{\langle S_z\rangle}{\langle J_z\rangle} = \frac{\langle\vec{S}\cdot\vec{J}\rangle}{\langle\vec{J}\cdot\vec{J}\rangle}$$

$$E' = \langle njm|\, L_z + 2S_z\,|njm\rangle.$$

$$= m\hbar + \langle njm|\, S_z\,|njm\rangle$$

i.e. choose from which S that lifts this degeneracy.

$$= m\hbar\left(1 + \frac{\langle\vec{S}\cdot\vec{J}\rangle}{\langle\vec{J}\cdot\vec{J}\rangle}\right)$$

$$= m\hbar\left(1 + \frac{j(j+1) - \ell(\ell+1) + 3/4}{2j(j+1)}\right) = \begin{cases} 1 + \frac{1}{2\ell+1} & j = \ell + 1/2 \\ 1 + \frac{1}{2\ell-1} & j = \ell - 1/2 \end{cases}$$

remark: gauge choice is not important because the leading term is $\vec{\mu}\cdot\vec{B}$ not relying on $\vec{A}$. and. $\vec{A}^2$ term which give you B^2 term is (x^2+y^2). which is a higher order perturbation. but doesn't influence the results up to now, if

9. Paschen - back effect

$$H = \frac{p^2}{2m} - \frac{e^2}{r} + \alpha_{L\cdot S}\,\vec{L}\cdot\vec{S} - \frac{eB}{2mc}(L_z + 2S_z)$$

Zeeman: $\alpha^2\frac{e^2}{a}$ $\quad\frac{e^2}{a}\frac{\phi}{\Phi_0}$ $\quad\frac{\phi}{\phi_0} << \alpha_{L\cdot S}$

Paschen - back: $\quad\frac{\phi}{\phi_0} >> \alpha_{L\cdot S}$

$\langle n l s \, l_z' \, s_z' | \cancel{\ } L_z + 2S_z | n l s \, l_z \, s_z \rangle$

$= \delta_{l_z l_z'} \delta_{s_z s_z'} \cdot (l_z + 2s_z)$

degeneracy: $\Delta E(l_z+2, -\frac{1}{2}) = \Delta E(l_z, \frac{1}{2})$

degenerate Stark shift:

$$\langle n l s \, l_z' \, s_z' | \underset{\alpha_{L\cdot S}}{\ } \vec{L}\cdot\vec{S} | n l s \, l_z \, s_z \rangle$$

$$= \langle n l s \, l_z' \, s_z' | L_z S_z + \frac{L_+ S_- + L_- S_+}{2} | n l s \, l_z \, s_z \rangle$$

$$= \langle \alpha_{\vec{L}\cdot\vec{S}} \rangle \cdot l_z S_z \cdot \delta_{l_z l_z'} \delta_{s_z s_z'}$$

remark: $H = \frac{p^2}{2m} - \frac{e^2}{r} + \alpha_{\vec{L}\cdot\vec{S}} \vec{L}\cdot\vec{S} - \frac{eB}{2mc}(L_z + 2S_z) + \frac{B^2}{2mc^2}(X^2+Y^2)$.

order: $\frac{e^2}{a}$ $\quad \alpha_2^2 \frac{e^2}{a}$ $\quad \frac{e^2}{a}\left(\frac{\phi}{\phi_0}\right)$ $\quad \frac{e^2}{a}\left(\frac{\phi}{\phi_0}\right)^2$

V. time dependent P.T. (perturbation theory).

1 — $H = H_0 + V(t)$

$i\hbar \partial_t |\psi(t)\rangle = (H_0 + V)|\psi(t)\rangle$

def (1) define: $|\psi(t)\rangle = e^{-iH_0 t/\hbar} |\alpha(t)\rangle$

$\Rightarrow \quad i\hbar \partial_t |\alpha(t)\rangle = e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar} |\alpha(t)\rangle$

form (1). $\quad H_0 |n\rangle = E_n |n\rangle$

try to solve: $|\alpha(t)\rangle \overset{\Delta}{=} \sum C_n(t)|n\rangle$

$\Longrightarrow i\hbar \dot{C}_m = \sum_n e^{iE_m t/\hbar} \langle m|V|n\rangle e^{-iE_n t/\hbar} C_n(t)$

2 — . 2×2 exact problem

$H = H_0 + V = \begin{pmatrix} E_1 & \\ & E_2 \end{pmatrix} + \begin{pmatrix} 0 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & 0 \end{pmatrix}$

$\Rightarrow i\hbar \begin{pmatrix} \dot{C}_1 \\ \dot{C}_2 \end{pmatrix} = \begin{pmatrix} 0 & \gamma e^{i\omega t} e^{-i\omega_{21} t} \\ \gamma e^{-i\omega t} e^{i\omega_{21} t} & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \qquad \omega - \omega_{21} \overset{\Delta}{=} \Delta.$

form (2) $\underline{c_1} = e^{-i\frac{1}{2}\Delta t} C_1 \qquad \underline{c_2} = e^{i\frac{1}{2}\Delta t} C_2.$

$|A_{b_1}| = \frac{\gamma}{2\sqrt{\alpha^2+\gamma^2}} \quad \alpha = \frac{\hbar}{2}\Delta$

$\Rightarrow i\hbar \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2}\Delta & \gamma \\ \gamma & -\frac{\hbar}{2}\Delta \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ now time independent!

you could easily ~~diag~~ diagnalize it. $\quad P = |c_2|^2 \overset{\text{initial on } 1, \delta}{=\!=\!=} 4|A_{b_1}|^2 \sin^2\frac{\lambda_+ t}{2\hbar}$

$\lambda_\pm = 2\sqrt{\alpha^2+\gamma^2}$

16.

remark: $\vec{a} \triangleq P\vec{b}$. then.
$i\hbar\dot{\vec{a}} = C\vec{a} \Leftrightarrow i\hbar(\dot{P\vec{b}}) = CP\vec{b} = i\hbar \cancel{(\partial_t} P(\partial_t + (\dot{\ln P}))\vec{b}$
$\Leftrightarrow i\hbar(\partial_t + (\dot{\ln P}))\vec{b} = P^{-1}CP\vec{b} \triangleq D\vec{b}$

$i\hbar\partial_t\rho = [\rho, H] = [\rho, H_0+V]$.
$\Rightarrow i\hbar\partial_t\langle\psi|\rho\rangle = \langle \overbrace{[\rho, H_0]}^{0} + [\rho, V]\rangle$.
$= \langle[\rho, V]\rangle =$ ~~...~~

Lorentzian profile, no damping, $H^\dagger = H$ then probability conserved ?

3- perturbative analysis

use the def (1) in 1- . i.e. the Heisenberg interaction picture.

formal solution:

form(1) $|\psi(t)\rangle_I = T e^{-\frac{i}{\hbar}\int_0^t H_I d\tau}|\psi(0)\rangle$

give you the "Taylor's" expansion, we could deal with different H_I order.
sort of "perturbation".

$$T e^{-\frac{i}{\hbar}\int_0^t H_I d\tau} = 1 + (-\frac{i}{\hbar})\int_0^t H_I d\tau + (-\frac{i}{\hbar})^2\int_0^t\int_0^{t_1} dt_1 dt_2\, H_I(t_1)H_I(t_2) + \dots$$

4- Fermi-Golden Rule.

ass (1). $V(t) = V\cdot\Theta(t-t_0)$ $(t_0 \equiv 0)$ step function

form (1). $P_{ni}(t) = |C_n(t)|^2$

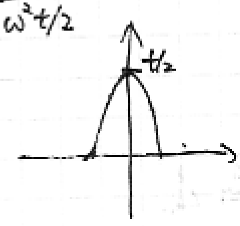
$|\psi(t)\rangle_I \approx |i\rangle + (-\frac{i}{\hbar})\int_0^t H_I(t)|i\rangle$.

$|\psi(t)\rangle_I \approx \sum C_n e^{-iE_nt/\hbar}$

$\Rightarrow |c_n|^2 = |(-\frac{i}{\hbar})\int_0^t \langle n|H_I|i\rangle|^2 = 4\frac{|V_{ni}|^2}{\hbar^2}\sin^2(\frac{\omega_{ni}t}{2})/\omega_{ni}^2$

for $t \lesssim \frac{\pi}{\omega_{ni}}$ $\Rightarrow$ $P_{i\to n}(t) \sim t\cdot\frac{\pi}{\omega_{ni}} \approx (t\cdot\Delta E)$.

$\frac{\sin^2}{\omega^2 t/2}$



trick? $\sum P_{i\to n}(t) \approx \int_{E_n\approx E_i} \frac{|V_{ni}|^2}{\hbar^2}\cdot\frac{4\sin^2(\omega_{ni}t/2)}{\omega_{ni}^2}\rho(E_n)\,dE_n$

$\approx \frac{|V_{ni}|^2}{\hbar^2}\cdot 2t\cdot\int_{n\approx i}\frac{\sin^2(\omega_{ni}t/2)}{\omega_{ni}^2 t}\rho(E_n)\,dE_n$

as $t\to\infty$, we have: $\frac{\sin^2(\omega_{ni}t/2)}{\omega_{ni}^2 t/2} \to \pi\delta(\omega_{ni})$

☆ rate of transition $\triangleq \frac{|C_n|^2}{t} \approx \frac{2}{\hbar^2}|V_{ni}|^2\cdot\pi\cdot\delta(\omega_{ni})$

this is the so-called Fermi-Golden Rule.

2017/3/28.

5 - Harmonic Interaction.

form (1) $V(t) = V e^{i\omega t} + V^{+} e^{-i\omega t}$

$$C_n \approx \left(-\frac{i}{\hbar}\right)\left[V_{ni} \cdot \frac{2\sin\left((\omega_{ni}+\omega)\frac{t}{2}\right)}{\omega_{ni}+\omega} \cdot e^{i\frac{t}{2}(\omega_{ni}+\omega)} + V^{+}_{ni} \cdot \frac{2\sin\left((\omega_{ni}-\omega)\frac{t}{2}\right)}{\omega_{ni}-\omega} \cdot e^{i\frac{t}{2}(\omega_{ni}-\omega)}\right]$$

$$\overset{?}{\Longrightarrow} R(t) = \left(\frac{2\pi}{\hbar}\right)|V_{ni}|^2 \{\delta(E_{ni}-\hbar\omega) + \delta(E_{ni}+\hbar\omega)\}$$ { absorption / stimulated emission

6 - Atom interaction with EM wave.

form (1). $H = H_0 - \frac{e}{2mc}(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}) + \frac{e^2}{2mc^2}A^2$

form (2). $\nabla\cdot\vec{A} = \vec{p}\cdot\vec{A} = 0.$ radiation gauge.

one "solution" could be : $\begin{cases}\vec{A} = A_0\vec{\varepsilon}\, 2\cos(\vec{k}\vec{x}-\omega t)\\ \vec{\varepsilon}\cdot\vec{k} = 0.\end{cases}$

Case (1). 7 - excited ~~states~~ atoms.

$$\begin{cases} R(t) \cong \frac{2\pi}{\hbar}|V_{ni}|^2 \cdot \delta(E_n - E_i - \hbar\omega). \\ V_{ni} = \langle n|\left(-\frac{e}{mc}\right)A_0 e^{-i\vec{k}\cdot\vec{x}}\,\vec{\varepsilon}\cdot\vec{p}\,|i\rangle \end{cases}$$

ass (1). $\lambda >> a_0$. i.e. $\omega << \omega_{\text{somewhat}}$.

$$V_{ni} \cong -\frac{eA_0}{mc}\langle n|\vec{\varepsilon}\cdot\vec{p}|i\rangle \cdot e^{i\cdot\cos\omega t}$$

cross - section : $\sigma_{i\to n} \triangleq \dfrac{\text{Power absorbed}}{\frac{\text{Power in}}{\text{Area}}} = \dfrac{R(t)\,\hbar\omega}{\langle\frac{E^2+B^2}{8\pi}\rangle\cdot c} = \dfrac{R(t)\,\hbar\omega}{\frac{c\cdot k^2 A_0^2}{2\pi}}$

$$\sigma_{i\to n} = \frac{2\pi}{m\omega}^2\left(\frac{e^2}{\hbar c}\right)|\langle n|\vec{\varepsilon}\cdot\vec{p}|i\rangle|^2\,\hbar\omega\,\delta(E_n - E_i - \hbar\omega).$$

form (3) and $\langle n|\vec{\varepsilon}\cdot\vec{p}|i\rangle = \frac{im}{\hbar}\langle n|[H_0, \vec{\varepsilon}\cdot\vec{x}]|i\rangle = \frac{im}{\hbar}E_{ni}\langle\vec{n}|\vec{\varepsilon}\cdot\vec{x}|i\rangle.$

$$\Rightarrow \sigma_{i\to n} = 4\pi^2\alpha_E|\langle\vec{n}|\vec{\varepsilon}\cdot\vec{x}|i\rangle|^2\,\delta(E_n - E_i - \hbar\omega)\cdot\hbar\omega.$$

$\vec{\varepsilon}\cdot\vec{X} = \varepsilon_x X + \varepsilon_y y. = \alpha Y_1^1 + \beta Y_1^{-1}.$ $\Rightarrow$ transition rule!

$M_n = M_i \pm 1.$ $\Pi_n = (-1)\Pi_i$

Case (2). 8 - ionization : bound $\to$ continuum.

$\delta \Rightarrow \frac{dN}{dE_n} = \frac{V}{\pi^2}\frac{m}{\hbar^2}k_F$ $E_n = \frac{\hbar^2 k_F^2}{2m}$ $N = 2\cdot V\cdot\frac{4\pi}{(2\pi)^3}\cdot\frac{k_F^3}{3}$

$\frac{d\sigma}{d\Omega}$ = partial differential cross section

$$= \left(\frac{2\pi}{\hbar}\right)|\langle n|e^{-i\vec{k}\cdot\vec{x}}\left(-\frac{e}{mc}\vec{\varepsilon}\cdot\vec{p}\right)|i\rangle|^2 \cdot \frac{V}{\pi^2}\cdot\frac{mk_F}{\hbar^2}\cdot\frac{1}{4\pi}$$

$$= \left(\frac{2\pi}{\hbar}\right)\left(\frac{e}{mc}\right)^2|\vec{\varepsilon}\cdot(\hbar\vec{k}_f - \hbar\vec{k})|^2\cdot N^2\cdot\frac{mk_F}{4\pi\hbar^2}\cdot\left|\int d\vec{x}\, e^{-i\vec{q}\cdot\vec{x} - \frac{2r}{a_0}}\right|^2$$

18.

LACK. variational method. (2017-3-30)

2017-4-4.

2 - permutation operator:

$P_{12}|12\cdots n\rangle = |21\cdots n\rangle \quad P_{12}^2 = I. \quad [H, P_{12}] = 0.$

3 - statistics (indistinguishable)

degeneracy $[H, P_{12}] = 0 \quad \{|\psi\rangle, P_{ij}|\psi\rangle\}$ give us. E.

selection rule: integer spin → bosonic → $P_{12} = +1$. symmetric wavefunction

half inte ······ fermionic ··· -1 anti-symmetric. ...

this is a rule determined by relativistic equation, Dirac equation, cannot show this from ~~the~~ the Hamiltonian right now.

4 - example:

2 particle: $\psi = \frac{1}{\sqrt{2!}}\{\phi_1(x_1)\phi_2(x_2) \pm \phi_1(x_2)\phi_2(x_1)\}$.

N particle: $\psi = \frac{1}{\sqrt{N!}}\sum_{perm}\prod \phi_i(x_{P_i})$ Boson.

$\begin{vmatrix} \phi_1(x_1) & \phi_1(x_2) & \cdots \\ \phi_2(x_1) & \cdots & \\ \vdots & & \end{vmatrix}$ fermion.

5 - Induced quantum interaction.

2 particle: $|\psi|^2 = \frac{1}{2}\int\{2|\phi_1|^2|\phi_2|^2 \pm \phi_1(x_1)\phi_2(x_2)\phi_1^*(x_2)\phi_2^*(x_1)$
$\pm \phi_1(x_2)\phi_2(x_1)\phi_1^*(x_1)\phi_2^*(x_2)\} dx_1 dx_2$

yet if we look deeper. $\psi(x_1, x_2)$.

then $|\psi(x_1, x_2)|^2 = \begin{cases} 2|\phi_1(x_1)|^2|\phi_2(x_1)|^2. \\ 0. \end{cases}$ ~~fix~~ Boson enhancement / fermion. repulsion

remark: the only case we need to pay attention is if we have $\phi_1 = \phi_2$, which would be not so easy to make a correction

6 - 2e system $\psi(1,2) = \phi(x_1, x_2)\chi(s_1, s_2)$.

$\psi_{S=0}(1,2) = \frac{1}{\sqrt{2!}}\{\phi_1(x_1)\phi_2(x_2) + \phi_2(x_1)\phi_1(x_2)\} \otimes \frac{1}{\sqrt{2!}}\{|+-\rangle - |-+\rangle\}. \quad \times 1.$

$\psi_{S=1}(1,2) = \frac{1}{\sqrt{2!}}\{\phi_1(x_1)\phi_2(x_2) - \phi_2(x_1)\phi_1(x_2)\} \otimes \frac{1}{\sqrt{2!}}\{|+-\rangle + |-+\rangle\} \quad \times 3.$

or $|++\rangle$

or $|--\rangle$.

7 - Young Tableaux

remark: the principle here is: for boson, this is easy to construct. because, they are all "plus". but for fermion. we don't need χ to be totally antisymmetric, but part symmetric, yet the relating spatial part to be ~~sym~~ antisymmetric to the rest cases!

(1)

~~A~~. for 1/2: ① [1], [2] ≡ [↑], [↓] ≡ [+], [−].

□ × □ = □□ + ⊟

②. □□ ≡ [1|1], [2|2], [1|2]. ~~J~~ $\Rightarrow$ 0. $J = (1\times 2+1 = 3)$

③ ⊟ ≡ [1/2] > $J = 1$. $J=0$.

④ (□ × □) × □ = (□□ + ⊟) × □ = ⊞ + □□□ + ⊞ + ⊟ redundancy / degeneracy in physics

always keep sth. in "left" or "up". otherwise ✗.

but for our case 3 is forbidden, thus ⊟ ✗

⊞ = [1|1 / 2] + [1|2 / 2] $S = 1/2$

□□□ = 111, 112, 122, 222. $S = 3/2$

discussion: these diagrams are only used to ~~c~~ count number of our system, after we write them out, we need to give them the correct symmetric / asymmetric arrangements.

(2). for spin 1: □ 1, 2, 3.

□ × □ = ⊟ + □□ = 3 + 6.

Counting rule:

$$\frac{3\times 4\times 2}{1\times 3\times 1} = 8 \quad \text{Mixed} \qquad \frac{3\times 4\times 5}{1\times 2\times 3} = 10 \quad \text{asymmetric?}$$

(3). for SU(N).

$$\square\times\square = \underset{A}{\boxminus} + \underset{S}{\square\square} = \frac{N\cdot(N-1)}{1\cdot 2} + \frac{N\cdot(N+1)}{1\cdot 2} = N^2$$

8- quark model.

Gell-man - Zweig: flavor quark: { u: up, d: down, s: strange. color: C red, blue, green }

Q_e		baryon B	isospin I_3	strangeness S	color
2/3	u:	1/3	+1/2	0	red, blue, green
−1/3	d:	1/3	−1/2	0	—
−1/3	s:	1/3	0	−1.	—

{ Q_e = electric charge = $I_3 + \frac{Y}{2}$; Y = hypercharge = $B + S$.

main goal is: although don't know Hamiltonian, first try to predict some symmetry terms from the ~~expression~~ behavior.

proton: uud.

$$[\Psi] = [X]\cdot[\text{spin}]\,[\text{flavor}] = [\psi_1\psi_2\psi_3]_S\,[S_u S_u S_d]_{A,S}\,[f_u f_u f_d]_{S,A}\ ?$$

20.

back to tableux:

$\square$ u, d, s $\longleftrightarrow$ 1, 2, 3.

Baryons qqq : $\square \times (\square \times \square) = \cdots\cdots$

among the expression, the term $[\square\square\square]$ has $\frac{3\times4\times5}{1\times2\times3} = 10$ terms.

Strange number / $I_3 \leftarrow$ (or Q_e?)							
0	uuu Δ^{++}		uud Δ^{+}		udd Δ^{0}		ddd Δ^{-}
1		uus Σ^{+}		uds Σ^{0}		dds Σ^{-}	
2			uss Ξ^{0}		dss Ξ^{-}		
3				sss Ω^{-}			

and p, n are refering to $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$ structure

9 — second-quantization (Dirac. Boson (§7), & Wigner-Jordan. Fermion (§8)).

① permanent state:

$|\beta_1 \cdots \beta_n\rangle \longleftrightarrow |N_1 \cdots N_\infty\rangle$ $\quad \sum N_i = N.$

slater det. $\quad$ occupancy

the algebra is the same as H.O. operators

fermion: $[C_\alpha, C_\beta^+]_+ = \delta_{\alpha\beta}$ $\quad N = \sum a_\alpha^+ a_\alpha$ or $\sum C_\alpha^+ C_\alpha$.

Bosons: $[a_\alpha, a_\beta^+] = \delta_{\alpha\beta}$.

?

$$|N_1 \cdots N_\infty\rangle \triangleq \frac{(\prod_\alpha N_\alpha!)^{1/2}}{(N!)^{1/2}} \sum_{[\beta]} |\cdots [\beta_1 \cdots \beta_{N_\alpha}] \cdots\rangle$$

② the expression of operators:

$H_0 = \sum h_i \quad h_i = \frac{p_i^2}{2m} + V(x_i). \quad H_1 = \frac{1}{2}\sum_{i \neq j} V(x_i, x_j)$

☆

$H_0 = \sum \langle\alpha|h|\beta\rangle a_\alpha^+ a_\beta$ (α and β refers to same state? i.e. $\delta_{\alpha\beta}$?)

the matrix element of H_0 !!!

and two body has:

$\therefore H_1 = \frac{1}{2}\sum_{\alpha\beta\gamma\delta} \langle\alpha\beta|V|\gamma\delta\rangle a_\alpha^+ a_\beta^+ a_\delta a_\gamma$

and for $\begin{cases} \text{Bosons} & [a_\alpha, a_\beta^+] = \delta_{\alpha\beta} \\ \text{fermions} & \{C_\alpha, C_\beta^+\} = \delta_{\alpha\beta} \end{cases}$

10 — perturbation theory.

Bosons: $H|N_0 \cdots N_\infty\rangle = \sum_\alpha (\overset{\circ}{E}_\alpha N_\alpha + \frac{1}{2}\langle\alpha\alpha|V|\alpha\alpha\rangle \cdot N_\alpha(N_\alpha - 1)) |N_0 \cdots N_\infty\rangle$

Fermions: $H|N_0 \cdots N_\infty\rangle = \sum_\alpha (\overset{\circ}{E}_\alpha N_\alpha + \frac{1}{2} V_{\alpha\beta,\alpha\beta} N_\alpha N_\beta - \frac{1}{2} V_{\alpha\beta,\beta\alpha} N_\beta N_\alpha) |N_0 \cdots N_\infty\rangle$

11. Hartree - Fock Approximation.

$\left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi_\alpha = E^0_\alpha \psi_\alpha. \Rightarrow E^0_\alpha = \int \psi^*_\alpha \left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi_\alpha \, dx . N_\alpha$

?

$V_{\alpha\beta \cdot \alpha\beta} = \int dx\, dx' \; \psi^+_\alpha(x)\, \psi^+_\beta(x')\, V(x,x')\, \psi_\alpha \psi_\beta \;\; N_\alpha N_\beta$

$V_{\alpha\beta \cdot \beta\alpha} = \int dx\, dx' \; \psi^+_\alpha(x)\, \psi^+_\beta(x')\, V(x,x')\, \psi_\beta \psi_\alpha . N_\alpha N_\beta$ functions !!

idea: we would seek $\{\psi_i(\vec{x})\}$ to get the optimised E^0_α. i.e.

$$\frac{\delta E_\alpha}{\delta \psi_\alpha} = 0. \qquad \text{Hartree - Fock equation}$$

$\Leftrightarrow \left[-\frac{\hbar^2}{2m}\nabla^2 + U\right]\psi_\alpha(\vec{x}) + \sum_\beta \int dx' \psi^+_\beta(x')\, V(x,x')\psi_\beta(x') N_\beta \;\; \psi_\alpha(x)$

$- \sum_\beta \int dx' \; \psi^+_\beta(x')\, V(x,x')\;\; \psi_\alpha(x')\, N_\beta \;\; \psi_\beta(\vec{x}). = 0.$

question: what's the meaning of $\alpha, \beta, \psi, N_\alpha$?

α refers to α-state, which ~~close~~ mix "single" particle.

thus in order to introduce statistics for particles in our Hamiltonian, we have:

$\left(-\frac{\hbar^2}{2m}\nabla^2 + U(x) + U_H(x)\right)\psi_\alpha(x) = 0$

$U_H(x) = \Sigma_\beta N_\beta \int dx' \psi^+_\beta(x')\, V(x,x')\, \psi_\beta$. "mean - field"

Thomas - Fermi Approximation:

for uniform ~~degeneracy~~ degenerate Fermi systems

Hartree: $\int dx' \; V(x,x') \, \underbrace{\left(\sum_\beta N_\beta \psi^+_\beta \psi_\beta\right)}_{} \;\; \approx \rho$

Hubbard Model:

$H = \Sigma\, \varepsilon_\alpha C^+_\alpha C_\alpha + \sum_{\langle i,j \rangle} J_{\alpha\beta}\, C^+_\alpha C_\beta$ (fermion !)

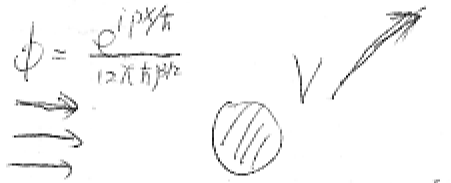
Anderson Model:

$H = H_{\text{Hubbard}}$ where ε_α is given randomly. $P(\varepsilon_\alpha) = e^{-\varepsilon_\alpha^2/2\sigma}$?

Anderson - localization.

IX Scattering

1. set - up $\quad \phi = \frac{e^{ipx/\hbar}}{(2\pi\hbar)^{3/2}}$ $\quad V \quad$ $\phi' = \frac{e^{ip'x/\hbar}}{(2\pi\hbar)^{3/2}}$



ellastic scattering: energy is conserved through this process. for incident particle only

time - dependent $\longleftrightarrow$ stationary problem

2. Lippman - Schwinger equation

$\begin{cases} \psi_+(t,\vec{x}) = \phi(t,\vec{x}) + \delta\psi_+(t,\vec{x}). \\ \psi_+(t,\vec{x}) \xrightarrow{t\to -\infty} \phi(t,\vec{x}) \end{cases}$ $\qquad \begin{cases} \phi(t,x) = \phi(x)\, e^{-i\varepsilon t/\hbar} \\ \delta\psi_+(t,x) = \delta\psi(x)\, e^{-i\varepsilon t/\hbar} \; e^{\varepsilon t} \; \lim_{\varepsilon \to 0^+} \end{cases}$

expansion in another base

22.

then schrödinger equation reads :

$$E[\phi + \delta\psi_+ e^{\varepsilon t}] + i\hbar\varepsilon\,\delta\psi_+ = (H_0+V)\psi_+$$

ass 0.

$$\Rightarrow \delta\psi_+ = \frac{1}{E-H_0+i\varepsilon}\delta V\psi_+ \qquad |\Delta\rangle = \frac{1}{E_0-H_0}(V+E_0-E)|\psi\rangle \quad E_0=E$$

$$\Rightarrow |\psi_+\rangle = |\phi\rangle + \frac{1}{E-H_0+i\varepsilon}V|\psi_+\rangle$$

L-S. equation.

$$\Rightarrow |\psi_-\rangle = |\phi'\rangle + \frac{1}{E-H_0-i\varepsilon}V|\psi_-\rangle$$

3- propagator.

$$G(E) \triangleq \frac{\hbar^2}{2m}\frac{1}{E-H_0+i\varepsilon} \xrightarrow[H_0=-\frac{\hbar^2\nabla^2}{2m}]{E=\frac{\hbar^2k^2}{2m}=\frac{\hbar k'^2}{2m}} \frac{1}{k^2+\nabla^2+i\varepsilon} \triangleq G(k) \qquad X\text{-rep.}$$

$$(k^2+\nabla^2)\langle x|G(k)|x'\rangle = \langle x|x'\rangle = \delta(x-x')$$

(Fourier transform + Cauchy integral

$$\langle x|G(k)|x'\rangle = G(k, x-x') = \frac{-1}{4\pi}\cdot\frac{e^{ik|x-x'|}}{|x-x'|}$$

$$\Rightarrow \psi_+(x) = \frac{e^{ikx}}{(2\pi)^{3/2}} + \frac{2M}{\hbar^2}\left(-\frac{1}{4\pi}\right)\int dx'^3\,\frac{1}{|x-x'|}e^{ik|x-x'|}V(x')\psi_+(x').$$

4 - Local approximation

$$|x-x'| = \sqrt{x^2+x'^2-2\vec{x}\cdot\vec{x}'} \qquad \vec{k}' = k\hat{x}$$

ass 1.

$$\Rightarrow \frac{e^{ik(x-x')}}{|x-x'|} \approx \frac{e^{ikx}}{x}e^{-i\vec{k}'\vec{x}'}$$

fourier transform !? convolution.

$$\Rightarrow \psi_+(x) = \cdots\cdots \triangleq \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\vec{x}} + f(\vec{k},\vec{k}')\frac{e^{ik|\vec{x}|}}{|\vec{x}|}\right].$$

we will call "f" to be scattering amplitude

$$f(\vec{k},\vec{k}') = \left(\frac{\hbar^2}{2m}\right)^{-1}\left(-\frac{1}{4\pi}\right)\cdot(2\pi)^{3/2}\int dx'\,e^{-i\vec{k}'\vec{x}'}\,V(\vec{x}')\,\psi_+(\vec{x}').$$

5 - ~~Bor~~ Born Approximation

ass 2. perturbation expansion from f to $\frac{e^{ikx}}{(2\pi)^{3/2}}$ forms. (V forms)

$$f^1(\vec{k}-\vec{k}'\triangleq\vec{q}) = \left(\frac{\hbar^2}{2m}\right)^{-1}\left(-\frac{1}{4\pi}\right)\underbrace{\int dx'\,e^{-i\vec{q}\vec{x}}\,V(\vec{x}')}\quad (=\tilde{V}(\vec{q}))$$

6 - Cross Section.

ass 3.

in-coming :

$$J_{in} = \phi^\dagger\vec{v}\phi = \frac{\hbar k}{m}(2\pi)^{-3}$$

$$J_{out} = \delta\psi_+^\dagger\vec{v}\,\delta\psi = \frac{\hbar k}{m}\frac{|f|^2}{(2\pi)^3|x|^2}$$

$$J_\Omega = |x|^2 J_{out}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} \triangleq \frac{J_\Omega}{J_{in}} = |f(\vec{k},\vec{k}')|^2$$

7- e.g.

$$V(\vec{x}) = A\,\delta(\vec{x}) \Rightarrow f = \frac{2M}{\hbar^2}\cdot\left(\frac{-1}{4\pi}\right)\cdot A = \text{const}$$

$$V(\vec{x}) = A\,\delta(r-\vec{a}) \Rightarrow f = \frac{2M}{\hbar^2}(-1)\cdot\frac{aA}{q}\sin qa. \quad \sim L \text{ dimension}$$

discussion: (1). $(\frac{\sin y}{y})^2$ has a beautiful ~~fo~~ shape.

(2). as $a \to 0$, we need to fix Aa^2 in order to give out the same meaning of A in the first example.

8 — T-matrix

rewrite the expression in 2 — we have:

$$V\psi_+ = V\phi + V\Delta_+ V\psi_+ \xleftrightarrow{T_+ = V\psi_+} T_+ = V\phi\rangle + V\Delta_+ T_+ \quad \text{(Attention)}$$

use $|\phi\rangle$ and $|\psi_+\rangle$ rather than ϕ and ψ^+. we have:

$|\psi\rangle = |k\rangle + \int f(k,k')\Delta(k')|k'\rangle dk'$

$$T_+ \overset{?}{=} \frac{1}{1 - V\Delta_+^*} V \qquad (T_+|\phi\rangle = |\psi_+\rangle ?)$$

$$= (1 + V\Delta_+^* + (V\Delta)^2 + \cdots) V \Rightarrow \boxed{f(\vec{k},\vec{k}') = \# \langle \vec{k}'|T|\vec{k}\rangle}$$

9 — optical theorem.

$$\boxed{V|\psi\rangle = \frac{1}{1-V\Delta} V|\phi\rangle = T|\phi\rangle}$$

$$\sigma_{TOT}^{(k)} = \text{total}\,\phi = \int d\Omega_{\vec{k}'} |f(\vec{k},\vec{k}')|^2$$

theorem: $\sigma_{TOT}(\vec{k}) = \frac{4\pi}{k} \operatorname{Im} f(\vec{k},\vec{k}')\Big|_{\vec{k}=\vec{k}'}$ probability, ~~QFT~~. loop and

LACK. T-matrix element derivation, Eixial-approximation (4-20-2017)

(4-25-2017).

12 — Partial wave equation. (everything could be written in bra ket !!)

$$f(k,k') = \frac{2m}{\hbar^2} \cdot \left(-\frac{1}{4\pi}\right) (2\pi)^3 \langle k'|T|k\rangle$$

dimension Coulomb in/out

$\langle k'|\,1\,\hat{T}\,1\,|k'\rangle$ expansion : use sperical bases

(1) where $\langle k'|E'l'm'\rangle \propto \beta_{el}\delta(E' - \frac{\hbar^2 k^2}{2m}) \cdot Y_{l'}^{m'}(\hat{k}')$

(2) and $\delta(E-)$ kill Σ_E, $\Sigma_{E'}$.

(3) $\langle E'l'm'|T|Elm\rangle \propto T(E)\,\delta_{ll'}\delta_{mm'}$ kill one of $\cdot\ \vec{k}\ l', m'$.

(4) δ_{m0} term kill m (where is this term?)

(5) finally we reach: $f(k,k') = \text{Const} \cdot \Sigma |\beta_{el}|^2 \cdot \left(\frac{2l+1}{4\pi}\right) \cdot T_l(E)\, P_l(\cos\theta)$.

$$= \Sigma\,(2l+1) \cdot \left(-\frac{\pi}{k}T_l\right) \cdot P_l(\cos\theta)$$

13 — optical theorem in this expansion

$$\begin{cases} \sigma_{tot} = \int d\Omega_k \frac{d\sigma}{d\Omega_k} \\ \sigma_{tot} = \frac{4\pi}{k} \operatorname{Im} f(k,k). \end{cases}$$

use $\int P_l(\cos\theta) P_{l'}(\cos\theta) = \delta_{ll'} \frac{4\pi}{2l+1}$

$P_l(1) = 1$.

we find: $\frac{\operatorname{Im} f_l}{k} = |f_l|^2$.

24.

14 - S-matrix

(1) $|\psi^+\rangle = |\vec{k}\rangle + \frac{1}{G(E)} V|\psi^+\rangle$.

$\Leftrightarrow$

$$\psi^+ = \phi(\vec{k}) + \frac{1}{\sqrt{(2\pi)^3}} \cdot f(k',k)\, e^{ikr}/r.$$

question: our incoming wave is a plane wave — Fourier transform

does it really matter in some cases?

do expansion to spherical directly:

$$\phi(\vec{k}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{i\vec{k}\vec{x}} = \frac{1}{\sqrt{(2\pi)^3}} \sum i^l J_l(kr) \cdot Y_l^{m}(\hat{k})\, Y_l^{m}(\hat{x}).$$

only S wave

$$\underline{\underline{\text{Taking } \hat{z}}} \quad \frac{1}{\sqrt{(2\pi)^3}} \sum i^l J_l(kr)\, Y_l^0(\hat{k})\, Y_l^0(\hat{x}) \quad (\delta_{m0}).$$

$$\Rightarrow \psi^+ = \frac{1}{(2\pi)^{3/2}} \cdot \sum (2l+1) P_l(\cos\theta) \left[i^l J_l(kr) - \frac{\pi T_l}{k} \frac{e^{ikr}}{r} \right]$$

$$= \times \quad \times \quad \times \cdot \left\{ \frac{e^{ikr}}{kr}\left[-\frac{i}{2} - \pi T_k\right] + \frac{e^{ikr}}{kr}(-1)^l \frac{i}{2} \right\}$$

trick on $r \to \infty$.

$$J_l(kr) \Rightarrow \frac{\cos(kr - (l+1)\frac{\pi}{2})}{kr}.$$

$$\begin{cases} S_l \triangleq 1 - i2\pi T_l \\ S \triangleq 1 - i2\pi T \end{cases}$$

(natural def. should be general?)

(2). the conservation of probability is a natural requirement. Tells me:

$$|S_l| = 1 \quad \Rightarrow \quad S_l \triangleq e^{2i\delta_l}$$

$$T_l = -\frac{1}{\pi} e^{i\delta_l} \sin\delta_l.$$

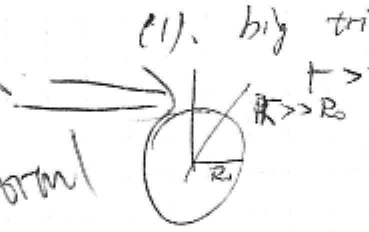
δ_l represent everything.

$$\sigma_{TOT} = \sum (2l+1) \frac{4\pi}{k^2} \sin^2\delta_l$$

$$f_l = -\frac{\pi}{k} T_l = \frac{1}{k} e^{i\delta_l} \sin\delta_l$$

15 - Phase-shift

(1). big trick: use Hankel ~~ex~~ function:



formal

$r \gg R_0$

$$\psi_l^{(+)}(r) = C_l^1 h_l^{(1)}(kr) + C_l^2 h_l^{(2)}(kr).$$

$$h_l^{(1)} \to \frac{e^{ikr - i(l+1)\frac{\pi}{2}}}{kr} \qquad h_l^{(2)} \to \frac{e^{-ikr + i(l+1)\frac{\pi}{2}}}{kr}.$$

$$h_l^{(1,2)} = J_l \pm i n_l \quad (\text{newmman}).$$

(2) $C_l^1 = S_l/2$. $C_l^2 = 1/2$. a phase shift: not an overall!

(3). procedure to find $\delta_l(E)$:

~~Inside~~: ψ_{in}.

out~~side~~: ~~ψ~~ $C_l^1 h_l^1 + C_l^2 h_l^2$.

match $r = R_0$: $\begin{cases} \partial \ln\psi = \text{const}. \\ \psi = \text{const} \end{cases}$ > coefficient!

16- hard core

$$e^{i2\delta_\ell} h_\ell^{i} + h_\ell^{e} = 0$$

$$\Rightarrow e^{i2\delta_\ell} = + \frac{-J_\ell + i n_\ell}{J_\ell + i n_\ell} \quad (\text{at } x = kR)$$

$$\Rightarrow \frac{J_\ell}{n_\ell} = \tan(\delta_\ell)$$

① $\ell = 0. \Rightarrow \delta_0 = -kR + n\pi.$ $\quad \frac{J_0}{n_0} \approx -\tan(kR).$

②. $kR \ll 1.$ and general ℓ :

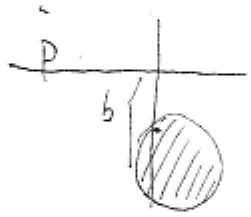
$$\begin{cases} J_\ell \to (kR)^{\ell} \\ \mathfrak{J} n_\ell \to (kR)^{-\ell-1} \end{cases}$$

$$\Rightarrow \delta_\ell \approx (kR)^{2\ell+1} \Rightarrow \sigma_{TOT} \approx \frac{4\pi \delta_0^2}{k^2} = 4\pi R^2.$$

?

③ $kR \approx 1$ or > 1 :

$$\tan\delta_\ell \approx \tan\left(kR + (\ell+1)\frac{\pi}{2}\right)$$



$$\sigma_{TOT} \to \sum (2\ell+1) \cdot \frac{4\pi}{k^2} \sin^2\left(kR + (\ell+1)\frac{\pi}{2}\right) \to \text{diverge !?}$$

$$\to \sum_{0}^{\ell_{max}} = 2\pi R^2.$$

$L = pb = \hbar k b$

$\Rightarrow \ell_{max} \approx kR$

17 - Phase shift redux

$$\psi_\ell^{(+)}(r) = e^{i\delta_\ell}(\cos\delta_\ell J_\ell - \sin\delta_\ell n_\ell).$$

put into equation : $\partial \ln\psi = \text{const.} \Rightarrow$

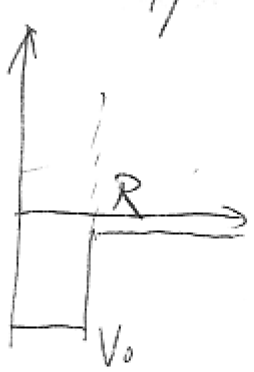
$$\underset{\text{(from (-) region)}}{\frac{\beta_\ell}{k}} = \frac{cJ_\ell' - s n_\ell'}{cJ_\ell - s n_\ell} \Rightarrow \tan\delta_\ell = \frac{xJ_\ell - J_\ell'}{x n_\ell - n_\ell'}$$

$$\frac{\beta_\ell}{k} \overset{\Delta}{=} x.$$

18- shadowing : just compare S_ℓ and T_ℓ, T_ℓ is infact kick away

S_ℓ's contribution from the outgoing incident wave $S_\ell = 1 + (-2\pi i T_\ell)$.

19 - Ramsauer Effect :



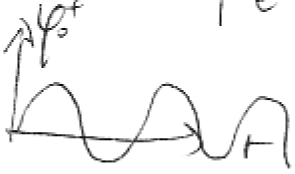
(1) $\ell = 0.$: $\psi_0^+ = e^{i\delta_0} \frac{\sin(kr + \delta_0)}{kr}$

$\boxed{\text{Where is } k\text{??}}$

let $\varphi_0^+ = r\psi_0^+$.

$$\Rightarrow \varphi_0^+ = \begin{cases} \sin(k'r). & E = \frac{\hbar^2 k'^2}{2m} + V_0 \quad r < R \\ e^{i\delta_0} \sin(kr + \delta_0) & E = \frac{\hbar^2 k^2}{2m} \quad r > R \end{cases}$$

original



$\to$ attractive potential

26.

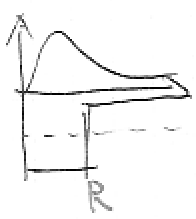
(2). $\begin{cases} \delta_0(k) = \alpha k \equiv -kR. \\ \sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \end{cases}$ choose $\lambda = 4R$. then:

$\Rightarrow \sigma_0 = \frac{4\pi}{k^2} = \frac{4R^2}{\pi}$

or choose $\lambda = 2R$. then $\sigma_0 = 0$.

20 - bound state and scattering length

$\begin{cases} a \triangleq \lim_{k\to 0^+} -\frac{\delta_0(k)}{k} \\ \text{i.e. } \delta_0(k) = -ak + \# k^2 + \cdots \end{cases}$ $\begin{cases} a<0 & \text{attractive} \\ a>0 & \text{repulsive} \end{cases}$



threshod bound state.

scattered: $\partial \ln \psi_0 = \frac{k}{\sin(kr+\delta_0)}$

bound : $\frac{\partial \ln\psi_0}{\partial r} = \varkappa$.

$\Rightarrow \varkappa \approx \frac{1}{a}$

$\Rightarrow \psi(r) \approx \frac{1}{r} e^{-r/a}$. $E_B = -\frac{\hbar^2\varkappa^2}{2m}$

$\psi_0(r) \sim \begin{cases} \sin(kr+\delta_0)/r & r<R \\ = \sin(kr)/r. & r>R. \end{cases}$

near the threshod bound state:

$\psi(r) \approx e^{-\kappa r}/r$. $E = -\frac{\hbar^2\varkappa^2}{2m} < 0$. $\frac{k}{\tan(kR+\delta_0)} = -\varkappa$

$\Rightarrow \tan\delta_0 \approx \frac{k}{\varkappa}$. $a \sim 1/\varkappa$.

21 - Analyticity of S-matrix

$l=0$. $\psi_0(r) = \# [\frac{e^{ikr}}{r} S_0(k) - \frac{e^{-ikr}}{r}]$

$k \to i\varkappa \xrightarrow{\text{observe}} \psi_0(r) = \# [\frac{e^{-\varkappa r}}{r} \infty - \frac{e^{\varkappa r}}{r}]$

complex analysis : $\begin{cases} \text{real part : amplitude analysis} \\ \text{imaginary part : correlation} \end{cases}$ 全纯.

$\Rightarrow S_0(k) = \frac{-k+i\varkappa}{k+i\varkappa} = e^{i2\delta_0}$ $\begin{cases} |S_0| = 1 \\ S_0 \to 1 \text{ as } k\to 0. \end{cases}$

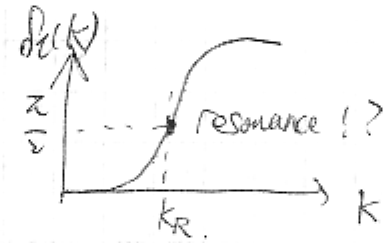
$\Rightarrow \sigma_0 = 4\pi|f_0|^2 = 4\pi|-\frac{\pi}{k}T_0|^2$

$= \frac{4\pi}{k^2+\varkappa^2}$

22 – resonances

$V_{eff} = \frac{\hbar^2 l(l+1)}{2mr^2} + V$

$E = \frac{\hbar^2 k^2}{2m}$



$\Rightarrow \cot(\delta_0) \approx 0 \approx \frac{E - E_R}{\Gamma/2} + O(E - E_R)^2$

$\sigma_{ToT} = \sum_l (2l+1)\, \sigma_l(k)$

★ $\sigma_l(E) = 4\pi |f_l(E)|^2 = \frac{4\pi}{k^2} \cancel{\sin^2\delta_l(E)} \frac{1}{\cot^2\delta_l + 1} \approx \frac{4\pi}{k_R^2} \cdot \frac{\Gamma^2/4}{(E-E_R)^2 + \Gamma^2/4}$

23 – Symmetries

Π – Invariance : $[\Pi, T^{(+)} = V + V\Delta^{(+)}T^{(+)}] = 0$.

$\Rightarrow$ in ~~the~~ plane-wave : $\langle k'|T|k\rangle = \langle k'|\Pi T \Pi^{-1}|k\rangle$.

$= \langle -k'|T|-k\rangle$

? Θ - reversal : $\langle \vec{k}'|T|\vec{k}\rangle = \langle \vec{k}'|T\vec{k}\rangle \overset{\Theta}{=} \langle \widetilde{T\vec{k}}|\tilde{\vec{k}}'\rangle$.

$= \langle \tilde{\vec{k}}|T^\dagger|\tilde{\vec{k}}'\rangle = \langle -\vec{k}|T^\dagger|-\vec{k}'\rangle$.

i.e. $f(\vec{k}', \vec{k}) = f(-\vec{k}, -\vec{k}')$

parity + Θ reversal :

$f(\vec{k}', \vec{k}) = f(\vec{k}, \vec{k}')$.

X. Dirac equation

1 – basic QM.

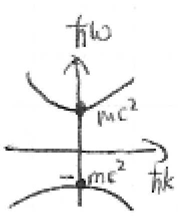
$i\partial_t|\psi\rangle = H|\psi\rangle$. $H = \frac{p^2}{2m} + \cdots$ $E = \langle H\rangle$

2 – relativistic QM. (~~Klein~~ Klein – Gordon).

try to recover : $E^2 = p^2c^2 + m^2c^4$.

just use the same form tells us :

$(\frac{1}{c^2}\partial_t^2 - \nabla^2 + m^2\frac{c^2}{\hbar^2})|\psi\rangle = 0$. ~~(... $|\psi\rangle$ because I use ∇ already)~~



$\Rightarrow$ Klein – Gordon : $\hbar\omega = \pm\sqrt{(\hbar kc)^2 + m^2c^4}$ Yet fails in $L\cdot S$ coupling

minus sign refers to anti-particle.

3 – . Dirac equation

$\psi \to \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{bmatrix}$ $H \to H_{Dirac} = 4\times4$

$E^2 = H_D^2 = (p^2c^2 + m^2c^4)\, I_{4\times4}$

use form $H = \vec{\alpha}\cdot\vec{p}\, c + \beta mc^2$, $\alpha_i^\dagger = \alpha_i$, $\beta^\dagger = \beta$.

NEXT FRIDAY 11:30 — 1:50.

28.

then use equation for E^2. we find that:

$$\cancel{[\alpha_i, \beta]}\ [\alpha_i, \alpha_j]_+ = 2\delta_{ij}.$$

where $\alpha_0 \triangleq \beta$. and $\alpha_i^+ = \alpha_i$.

3.5 ~~4~~ — base - relation.

symmetry concern : $U(1)$ symmetry for the non-relativistic case.

$U(N)$ symmetry for this relativistic case.

Dirac - basis : $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$. $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ✓

4 — application to hydrogen atom

Dirac : $\begin{cases} i\partial_t \to i\partial_t - eA_0 \\ \vec{p} \to \vec{p} - \frac{e}{c}\vec{A} \end{cases}$

then : $(i\partial_t - eA_0)\psi = H_D \psi$.

$\Rightarrow$ spin - orbit with additional $1/2$. and $g_e = 1/2$. fixed

Casimir fluctuation ; ? vacuum field fluctuation?