HW: tonic.physics.stonybrook.edu/~syorisya/phy540_2017/hw/ ... pdf

SM I microscopic parameters (CM. EM. QM) - 1/23/2017

(real thermo dynamics)

1. $\frac{\sqrt{\delta E^2}}{\bar{E}} = \frac{1}{\sqrt{N}} \cdot \frac{\sqrt{\delta E_1^2}}{E_1}$ $E = \Sigma E_i$ $\overline{(E-\bar{E})^2} \overset{\Delta}{=} \delta E^2$

2. ① $\tau_{observe} >> \tau_{relax}$ $\Rightarrow$ system always in equilibrium

$\tau_{change} >> \tau_{relax}$ $\Rightarrow$ slow change, stay in equilibrium

② $\tau_{relax} \gtrsim \tau_{change} >> \tau_{relax, subsys}$ local equilibrium

3. First Law: energy conservation

$$\Delta E = \delta W + \delta Q$$

Second Law: disorder grows spontaneously, reach some maximum under fixed condition

Third Law: disorder $\to 0$ as $T \to 0$.

remarks: like ideal gas is not $C_V = \frac{3}{2}$ anymore at $T=0$, QM takes account! we could deal with some situations with the third law.

4. expression of dW.

CM: $dW = -p\,dV + \sigma \cdot dA + f \cdot dL$

EM: $dW = \vec{E} \cdot d\vec{P} + \vec{H} \cdot d\vec{M}$

5. isolated system:

V. P. M. A. L to be constant.

6. S = measure of disorder.

① hope. $S = S_A + S_B$. {A} $\not\leftrightarrow$ {B}

② $\frac{d}{dt} S \geq 0$ (2nd Law). $S \xrightarrow{time} S(E, V)$

1/25/2017

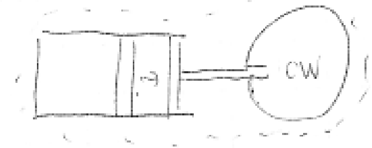
7. entropy. $S \overset{\Delta}{=} \cancel{=} \max_{\Sigma E_i = E} \Sigma S_i(E_i, V_i)$

① $(\Delta S)_{V_i} = (\Delta E_2) \cdot (-\frac{\partial S_1}{\partial E_1}) + (\frac{\partial S_2}{\partial E_2}) \geq 0 \Rightarrow T_1 \geq T_2$ (remember $\frac{\partial S}{\partial E}|_V = \frac{1}{T}$

sounds like S is also like V. M. A. L., the status quantity!

rather than force $\cancel{=}$ P. $\vec{B}$. $\vec{E}$. σ. f ...

② $\Delta S = \cancel{=}0$. S = const.



isolated.

$\Delta S \cancel{= \Delta S_{cw}} = 0$. $\Delta S_{total} = \Delta S_{cw} \geq 0$

$\Delta E = -p\,dV$

$\Rightarrow p \geq p_{cw}$

acceleration $\Rightarrow S_{total} > 0$

2.

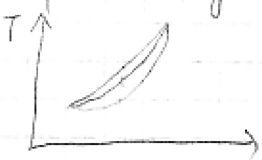
Finally we reach this important expression:

$$dE = TdS - pdV. \qquad \text{for reversible process.}$$

$$\frac{\delta Q}{dT} = \frac{(dE + pdV)}{dt} \le \frac{TdS}{dt}. \qquad \text{for irreversible process.}$$

and equilibrium only reaches when it is reversible.

(what the fuck is this graph? T-S?)



$\oint TdS \neq 0$. for the ~~circle~~ line.

$\oint_{\text{forward time}} \frac{\delta Q}{T} \le \oint dS = 0.$ for the circle

③ Some cases:

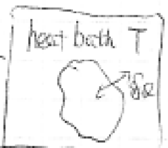
$V = \text{const}.$ $\quad C_V = (\frac{\partial E}{\partial T})_V$

$p = \text{const}.$ $\quad C_p = \frac{\delta Q}{dT} = (\frac{\partial H}{\partial T})_p$ $\quad (H = E + pV)$

remark: although with a fixed loop, the parameters like $V, p, n, \vec{B}, \vec{E}, \dots$ get back to the original value, yet the inner energy varies; and among the reasons of this case, the most important thing is $\frac{\delta Q}{T} = dS$ for reversible process is a status function which directly connects δQ and energy term.

$S = \text{const}$

$T = \text{const}$



1) $(\Delta E)_S = -pdV = (\delta W)_{ext}$.

2) $(\delta W)_{ext} = \Delta(E - \cancel{pV}\,TS) \triangleq \Delta F$ $\qquad F = E - TS$.

remark: if you want to fix S or T, then add a heat bath.

and finally $T = \text{const}$, $p = \text{const}$

$$G \triangleq E + pV - TS \qquad (\text{yet here } \delta Q \ne \Delta G.\ \text{balabala}).$$

and also G has another general meaning of chemical energy? number energy?

$$G = n\mu(T, P)$$

guess: if the universe doesn't allow $\delta W = pdV$ or something alike, then there won't be such colorful world, then for mathematics, it is referring to multi-variables

Suggestion: make sense of those physical terms of basis meaning!

1/27/2017

8. ① Maxwell's relation:

$dE = TdS - pdV$ $\qquad dH = TdS + Vdp$.

$dF = -SdT - pdV$ $\qquad dG = -SdT + pdV$.

......

② and we define:

$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$ thermal expansion coefficient.

$K_T = \frac{1}{V}\left(-\left(\frac{\partial V}{\partial p}\right)_T\right) > 0$ isomal

$K_S = \frac{1}{V}\left(-\left(\frac{\partial V}{\partial p}\right)_S\right) > 0$ adiabatic

compressibility

$C_p = \left(\frac{\partial H}{\partial T}\right)_p$

$C_V = \left(\frac{\partial E}{\partial T}\right)_V$.

exercise 1:

$\left(\frac{\partial H}{\partial p}\right)_T \overset{\text{ideal gas}}{=\!=} 0$. $\quad \left(\frac{\partial V}{\partial T}\right)_p = N/p \quad \alpha = 1/T$.

③ Some math tricks that you might use:

(1). $df = (\partial_x f)dx + (\partial_y f)dy$.

(2). $(\partial_x f)_y = 1/(\partial_f x)_y$

(3). $(\partial_x f)_y \cdot (\partial_y x)_f \cdot (\partial_f y)_x = -1$.

$\boxed{\text{easy way to express}}$

(4). $\frac{\partial(f, g)}{\partial(x, y)} = \frac{\partial(f, g)}{\cancel{\partial(x,y)}} \cdot \frac{\partial(P, Q)}{\partial(X, Y)} \cdot \frac{\partial(X, Y)}{\partial(x, y)}$. $\qquad \frac{\partial(a, b)}{\partial(c, d)} = \det\begin{pmatrix} \partial_c a & \partial_c b \\ \partial_d a & \partial_d b \end{pmatrix}$.

$\frac{\partial(f, y)}{\partial(x, y)} = (\partial_x f)_y$. $\qquad \frac{\partial(X_1)}{\partial(X_2)} \cdot \frac{\partial(X_3)}{\partial(X_4)} = \frac{\partial(X_1)}{\partial(X_4)} \cdot \frac{\partial(X_4)}{\partial(X_2)} \cdot \frac{\partial(X_3)}{\partial(X_6)} \cdot \frac{\partial(X_6)}{\partial(X_4)}$

$= \frac{\partial(X_1)}{\partial(X_4)} \cdot \frac{\partial(X_3)}{\partial(X_6)}$.

e.g. 1. $K_T/K_S = C_p/C_V \triangleq \gamma$.

e.g. 2 $(\partial S/\partial V)_p = C_p/\alpha \cdot T \cdot V$

remark: so the main idea of this part is: I have enough variables, then I could express all of the differential relation by using them, which is used to help cancel the terms E. H. F. G. since they are status variables and ~~the~~ central physical terms. then we want to measure them by another way or see its physical meaning.

e.g. 3 $C_p - C_V = \alpha^2 \frac{TV}{K_T} > 0$.

General step: write out E. H. ...., then cancel them in some way. then everything is beautiful.

4.

e.g. 4. $\delta Q = dE + p\,dV = T\,dS = C_V\,dT + \frac{\alpha T}{\kappa_T}\,dV$

$= dH - V\,dp = C_p\,dT - \alpha T V\,dp$. (that's more fluent than e.g. 3).

☆ e.g. 5. ideal gas:

$S = N \cdot \lg(V/V_0) + S_0(T)$.

$C_V = T\cdot\left(\frac{\partial S}{\partial V}\right)_T = -T\left(\frac{\partial E}{\partial V^2}\right)_T$

$F = \int \frac{1}{T}\,dV \int dV \cdot (-C_V/T) + \cdots$

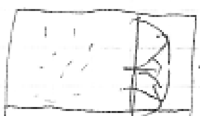
9. Joule experiment ①

$\delta Q = 0$

$H_1 = E_1 + p_1 V_1 = E_2 + p_2 V_2 = H_2$.

$\Rightarrow \left(\frac{\partial T}{\partial p}\right)_H = \frac{V}{C_p}[\alpha T - 1]$ $\left(\frac{\partial S}{\partial p}\right)_H = -\frac{V}{T} < 0$

②



isolated.

$E_1 = E_2$ (outer). (no force. although dV generally exist?!)

$\left(\frac{\partial T}{\partial V}\right)_E = -\frac{1}{C_V}\left[T\cdot\left(\frac{\partial p}{\partial T}\right)_V - p\right] \xrightarrow{\text{ideal gas}} 0$. (inner relation).

1/30/2017

10. other processes (know what we fix, know how to write the equation for process).

①

$\begin{cases} S \to \max \\ E = \text{const} \end{cases}$ or $\begin{cases} S = \text{const} \\ E \to \min \end{cases}$.

②

$T\Delta S \geq \delta Q \Rightarrow \Delta F \triangleq \Delta(E - TS) \leq 0$.

$F \to \min$. (T = const. V = const).

③ T = const p = const. $\Rightarrow$ $G \to \min$ $G \triangleq E - TS + pV$

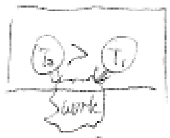
if $\begin{cases} \delta E \\ \delta V \\ \delta S \end{cases} \Rightarrow \delta G = \delta E + p\,\delta V - T\,\delta S$.

$= \left(\frac{\partial^2 E}{\partial S^2}\right)(\delta S)^2 + \left(\frac{\partial^2 E}{\partial V^2}\right)(\delta V)^2 + 2\frac{\partial^2 E}{\partial S\,\partial V}\cdot\delta S\,\delta V \geq 0$ (?)

$= \delta T\,\delta S - \delta p\,\delta V$.

main idea here: use the δ to estimate how it works for the equilibrium, which is well determined by the process already!!!

11. the thermo engine (heat transform).

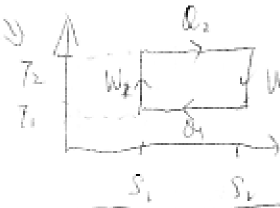


$\Delta W = E_{ini} - E_{fin}$ $\qquad S_{fin} \geq S_{ini}$

$\left(\frac{\partial(\Delta W)}{\partial S_{fin}}\right)_{isolated} = -\left(\frac{\partial E_{fin}}{\partial S_{fin}}\right)_V = -T_{fin} < 0.$

?

$\Delta Q = Q_2 - Q_1 = W_2 - W_1.$



$\eta \triangleq \frac{\Delta W}{Q_2} = 1 - \frac{T_1}{T_2}$

cooling $\longleftrightarrow$ heating.

| what's the main idea of this part? both physical meaning and the math assumption (condition)? |

2/1/2017.

12. def of chemical potential. $\mu \triangleq \frac{G(p,T)}{N}$ (or $G_{p,T}$) (not related to N for large enough system)

$\Rightarrow d\mu = -S_m dT + V_m dp$. thus we could rewrite the expressions as:

$dG = -SdT + Vdp + \mu dN$

$dA \to dA + \mu dN$. where $A = E, F, H$

then think of a process, when $V = \text{const}$. ~~$\mu = \text{const}$~~

① $d(F - \mu N) = -TdS - Nd\mu \triangleq d\Omega$.

Ω is called Grand potential $\stackrel{?}{=} F - G$.

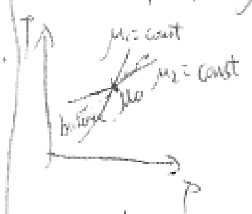
② $G = \mu_1 N_1 + \mu_2 N_2$ should be min at equilibrium

three situations: $\mu_1 > \mu_2$, $\mu_1 < \mu_2$, $\mu_1 = \mu_2$.

here μ_1 and μ_2 are const, while we are thinking of N changes.

thus $\mu = \mu(p, T)$ is absolute!!

③



~~$d\mu_1 = -S_m dT_1 + V_m dp_1$~~

$d\mu_1 = -S_{m1} dT + V_{m1} dp$

$d\mu_2 = -S_{m2} dT + V_{m2} dp$

equilibrium: $d\mu_1 = d\mu_2$

$\Rightarrow (S_{m1} - S_{m2})dT = (V_{m1} - V_{m2})dp$

i.e. $\left(\frac{\partial p}{\partial T}\right)_{\mu_1 = \mu_2} = \frac{S_{m1} - S_{m2}}{V_{m1} - V_{m2}} \triangleq \frac{\ell}{T(V_{m1} - V_{m2})}$.

| remark: know $f(T, p, V)$. i.e. state of equation + let $\mu_1 = \mu_2$ $\Downarrow$ $T = T(p)$. $V = V(p)$ all the variables come to 1. |

[Notice that in the equation, they are all intensity variables!]

④. e.g. with water :

evaporating $\rightarrow$ $p\uparrow$ as $T\uparrow$ $\Rightarrow$ $\frac{l}{V_1 - V_2} > 0$ $\Rightarrow$ $l > 0$. $S_1 > S_2$.

melting $\rightarrow$ $p\downarrow$ as $T\uparrow$ $\Rightarrow$ $\begin{cases} \Delta V < 0 \\ \Delta S > 0 \end{cases}$.

13. probability theory :

$\Delta W = \rho \cdot dp\,dq$. (density, population within p-q diagram)

ass(1) this ρ ~~is~~ $= \rho(p, q)$ when we reach the statistical equilibrium

then $\langle f \rangle = \int dp\,dq\, \rho \cdot f \,/ \int dp\,dq\, \rho$

or. $\bar{f} = \frac{1}{T} \int \rho(p,q,t) \cdot f(p,q)\, dp\,dq\,dt$.

2/3/17.

ass (2). $\rho_{stationary}$ is determined by integral of motion $(\vec{P}, \vec{J}, \underline{E})$

$\Rightarrow$ $\underline{\rho_{stat} = \rho(E(p,q))}$.

? e.g.1 micro - canonical distribution $\{E = E_0\}$

$\rho = \delta(E(p,q) - E_0)$. ergodicity.

no coefficient? ass (3). $|E_{m+1} - E_m| \ll \hbar/\Delta t$. (not enough time to balance).

~~e.g.2~~.

ass (4). $d\Gamma = \left(\frac{dp\,dq}{2\pi}\right)$. represent 1 state in ~~···~~ real world. thus we could calculate the quantity of states.

represent under density matrix :

$i\partial_t W = [H, W] = (E_m - E_n) W_{mn} \cdot |m\rangle\langle n| = 0$.

$\Rightarrow W_{mn} = W_m \delta_{mn}$ $\qquad W_m = \langle W \rangle_m$

e.g.2. as in ass (3), most of the time we could not tell a difference between these narrow band width. then $\rho(E) = $ Const

☆. e.g.3. N - ensembles (N systems). into M states.

different ways $= \frac{N!}{N_1! \cdots N_m!}$ $\qquad \{S\} \triangleq \lg\left(\frac{N!}{N_1! \cdots N_m!}\right) \approx \sum -N_i \lg \frac{N_i}{N}$

if we use Lagrange method, it is easy to get :

$F = -W_i \lg W_i - \lambda(\sum W_i - 1)$. $\Rightarrow$ $W_i = $ Const

gives you the minimum of F. which is ~~#~~ refering to the most occasional ~~#~~ distribution

2/6/2017.

① $N \gg 1$ identical QM oscillators, all states equal possibility.

$E^{(i)} = \hbar\omega\, m^{(i)}$

$E = \Sigma E^{(i)} = \hbar\omega \Sigma m^{(i)} = \hbar\omega M$.

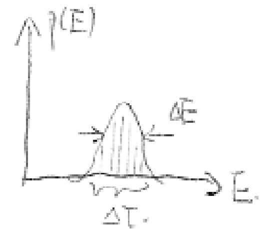
then for each M I have $\Omega = C_{M+N-1}^{M}$ "different" possible distributions.

$S = \lg \Omega = M \cdot \lg \frac{M+N-1}{M} + (N-1) \lg \frac{N+M-1}{N-1}$

$\overset{M \gg N}{\approx} N\,(1 + \lg (E/N)/\hbar\omega)$

$T = \left(\frac{\partial S}{\partial E}\right)^{-1} = E/N = \langle E_i \rangle$ $\left(\langle E_{kin} \rangle = \frac{1}{2} \langle E_i \rangle = T/2. \ \checkmark\right)$

②



$\Gamma(E)$ number of energy levels within $E_m \leq E$

$\Delta\Gamma = \frac{\partial \Gamma}{\partial E} \cdot \Delta E$

$S = \lg \Delta\Gamma$. and $\Delta\Gamma = e^S = \frac{\Delta E}{\delta E}$ ← width ← average gap

I am assuming that all energy levels are equidistant

$\Longrightarrow \delta E = \Delta E \cdot e^{-S} = \Delta E \cdot e^{-N \cdot \text{const}}$

③ full description of a system.

CM → all coordinate

QM → all $|m\rangle$. → microscopic description $\{p, q\}$ specific E_m $\Delta\Gamma = \frac{\partial \Gamma}{\partial E} \Delta E$

→ monoscopic Variables $\{E, V, \ldots\}$ $E - \frac{1}{2}\Delta E < E_m < E + \frac{1}{2}\Delta E$

→ $S = \lg \Delta\Gamma$

? Question

why ask these?

~~Question~~ Is the seperation arbitrary?

- $\Delta p \Delta q \to a \Delta p \Delta q$ elementary cell in phase space (QM)
  $S \to S + N \lg a$.
- particles distinguishable?
- QM tells use ambiguity.

14. canonical distribution

ass 1.

$E_\Sigma = E_1 + E_{HeatBath\,(=HB)} = \text{const !!!}$

$S_\Sigma = S_{HB} + S_1 \qquad \Delta S_\Sigma \geq 0$

$d\Gamma_\Sigma = \int d\Gamma_{HB}\, d\Gamma_1\, \delta(E_\Sigma - E - E_{Heatbath})$

ass 0. $\boxed{\frac{d\Gamma}{dE} = e^S/\Delta E. \quad \text{keep in mind!}}$

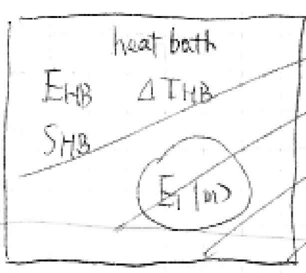
for 1 state $|m\rangle$ probabity to occupy is.

ass 2. $w_m = \left.\frac{d\Gamma_\Sigma}{d\Gamma_1}\right|_{E_\Sigma = E_1 + E_{HB}}$

$= \frac{1}{Z} e^{-\beta E_m}$

2/8/2017.

~~15 Gibbs dis distribution~~



$\cancel{\Delta\Gamma_{HB} = e^{S_{HB}}}$

15. partition function (related to F(T, V)!!)

$Z = \sum e^{-E_m/T}$

$\langle E\rangle = \frac{1}{Z}\sum E_m e^{-E_m/T} \underset{\beta = 1/T}{=} -\frac{\partial(\ln Z)}{\partial\beta}$

$C = \frac{\partial\langle E\rangle}{\partial T} = \beta^2\frac{\partial^2 \ln Z}{\partial\beta^2} = \beta^2\cdot\left[\frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2} - (\langle E\rangle)^2\right] = \beta^2[\langle E^2\rangle - \langle E\rangle^2].$

$= \beta^2(\delta E)^2$ ~~flutt~~ fluctuation !!

$\boxed{\text{remarks: } \delta E = T\sqrt{C}. \text{ since E and C are extensive variables. then } \delta E \sim \sqrt{N}. \text{ why this is so beautiful? any intuitive image?}}$

$\Rightarrow S = -\sum w_m \ln w_m = +\ln Z + E_m/T.$

$\Rightarrow F = E - TS = -T\ln Z\,(T, V).$

(and it is easy to verify that $S = -\left(\frac{\partial F}{\partial T}\right)_V$

e.g. 1. Quantum Oscillator. (throw away the $\frac{1}{2}\hbar\omega$ unit here!!!)

> Remark: F relies largely on the expression of Z, we could not just throw ~~a~~ "constant" away

$$Z = \sum_m e^{-\hbar\omega m/T} = \frac{1}{1-e^{-\hbar\omega/T}}$$

$$F = -T\ln Z = T\ln(1-e^{-\hbar\omega/T})$$

$$E = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = \hbar\omega/(e^{\hbar\omega/T}-1) \xrightarrow{T\to+\infty} T$$

$$\xrightarrow{T\to 0} \hbar\omega\, e^{-\hbar\omega/T} \simeq 0$$

$$C = \frac{\partial E}{\partial T} = \left[\frac{\hbar\omega}{2T}\cdot\frac{1}{\sinh(\frac{\hbar\omega}{2T})}\right]^2$$

e.g. 2. cavity photon (black body radiation)

$k = \frac{\pi}{nL}$ ☆

ass 1. $\underline{\text{number of photon}}$: $d\Gamma = \frac{V d^3p}{(2\pi\hbar)^3}\cdot 2$ (~~pt~~ polarization).

$$p = \hbar k = \hbar\frac{\omega}{c}$$

$$~~d\Gamma~~\ dN = V\frac{\hbar^2\omega^2\, d\omega\, d\Omega}{c^3(2\pi\hbar^3)}\cdot 2 \Rightarrow \underline{dN = V\cdot\frac{\omega^2 d\omega}{c^3\pi^2}}$$

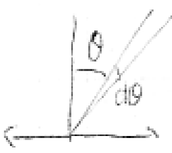
spectral density:

$$\frac{1}{V}\cdot\frac{dE}{d\omega} = \cancel{\hbar\omega}\ \frac{1}{V}\frac{dN}{d\omega}\cdot\frac{\hbar\omega}{e^{\hbar\omega/T}-1} = \frac{\hbar\omega^3}{\pi^2 c^3}\cdot\frac{1}{e^{\hbar\omega/T}-1} \triangleq u(\omega).$$

energy density:

$$\mathcal{E}(T) = \int_0^{+\infty} u(\omega)\,d\omega = \frac{\hbar}{\pi^2c^3}\cdot\left(\frac{T}{\hbar}\right)^4\int_0^{+\infty}\frac{x^3dx}{e^x-1}. \quad \left(=\frac{\pi^4}{15}\right).$$

2/10/2017.



an area, energy goes out:

$$\frac{dW}{dt} = \int_0^{\frac{\pi}{2}}\frac{2\pi\cos\theta\, d\theta}{4\pi}\cdot A\cos\theta\cdot c\cdot\mathcal{E} = \frac{1}{4}A\cdot c\cdot\mathcal{E} = \frac{\pi^2}{60c^2\hbar^3}\cdot T^4 \triangleq \sigma\cdot T^4$$

so-called Stefan-Boltzmann's Law.

for multi- ~~photon~~ ω level:

$$Z = \prod z_i. \qquad E = \cancel{\int}\sum \hbar\omega\cdot N(\omega).$$

$$F = -T\sum\ln z_i = -T\ \cancel{N(\omega)}\sum \cancel{\ln} N(\omega)\ \cancel{\ }\ln z(\omega)$$

$$= \int dN\cdot T\cdot\ln(1-e^{-\hbar\omega/kT}) = -\frac{E}{3} \propto VT^4.$$

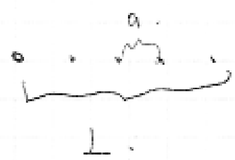
ass 2. ω allowed in the "chamber" is continuous $\Leftrightarrow$ L is not a fixed number, you can choose any L to give out the chamber size.

then we have: $\quad P = +\mathcal{E}/3$

$$G = 0 = \mu$$

10. e.

e.g. 3. phonons



ass(1) $\Delta x_n = \begin{cases} \cos(k_m \cdot n \cdot a) \\ \sin(k_m \cdot n \cdot a) \end{cases}$ $k_m = \frac{2\pi}{a \cdot L} \cdot m$ $-\frac{\pi}{a} < k_m < \frac{\pi}{a}$ Brillouin Zone

ass(2) $u = \omega/k$. (dispersion relation).

$\Downarrow$

$d\omega \cong u\, dk$. 2 transverse mode u_T 1 longitudinal mode u_L

ass(3). $\frac{4\pi k^2 dk}{(2\pi)^3} \approx \frac{4\pi \omega^2 d\omega}{(2\pi)^3} \underbrace{\left(\frac{1}{u_L^3} + \frac{2}{u_T^3} \right)}_{} \left(\approx \frac{3}{\bar{u}^3} \right)$.

☆ $\Downarrow$

$dN_\omega^{\text{photon}} = 3 \cdot V \cdot \frac{4\pi k^2 dk}{(2\pi)^3} = 3V \cdot \frac{4\pi \omega^2 d\omega}{(2\pi)^3 \bar{u}^3}$

ass(4) Debye frequency cut off:

number of modes $\underline{\underline{\text{should}}}$ $3 \cdot N_{atom}$ $\underline{\underline{\text{should}}}$ $\int_0^{\omega_D} d\omega \cdot \frac{dN}{d\omega} = 3 \cdot V \cdot \frac{\omega_D^3}{6\pi^2 \bar{u}^3}$

$\Rightarrow \omega_D = \left(\frac{6\pi^2}{V_m} \right)^{1/3} \cdot \bar{u} \Rightarrow \lambda_{max} = \frac{2\pi}{k} = \left(\frac{4}{3}\pi \right)^{1/3} \cdot V_m^{1/3} \underset{\sim\sim}{\overset{\Delta}{=} 1.6 \cdot a}$.

energy of virbration ~~at~~ at T.

ass(0). $E = \int_0^{\omega_D} dN_\omega < \varepsilon(\omega) >$ $\left(\hbar\omega_D \overset{\Delta}{=} T_D \right)$.

$= \frac{3V}{2\pi^2 \bar{u}^3} \left(\frac{T}{\hbar} \right)^4 \int_0^y \frac{x^3 dx}{e^x - 1} \Big|_{y = \hbar\omega_D / T} \cdot \left(\frac{\hbar\omega_D}{T} \right)^3$

Introduction of $\underline{\text{Debye's function}}$ $D(y) = \frac{3}{y^3} \int_0^y \frac{x^3 dx}{e^x - 1}$

ass(5). $y \to \infty$ $(T \to 0)$ $D(y) \to 1$

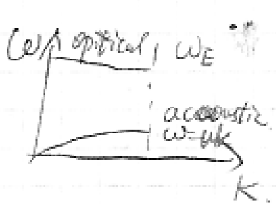
$y \to 0$ $(T \to \infty)$ $D(y) \to \frac{3}{y^3} \cdot \frac{\pi^4}{15} = \frac{\pi^4}{5y^3}$

$\Rightarrow E = 3 \cdot N_{atoms} \cdot T \cdot D(\hbar\omega_D / T) \begin{cases} \to 3NT & T \to \infty \\ \to 3NT \cdot \left(\frac{T}{T_D} \right)^3 \cdot \frac{\pi^4}{5} & T \to 0 \end{cases}$

? (look back!) $\Rightarrow C = \begin{cases} 3N & T \to \infty \\ \frac{12\pi^4}{5} N \left(\frac{T}{T_D} \right)^3 & T \to 0 \end{cases}$ $(\sim e^{-y}?)$.

ass(4') Einstein's model:

all frequency $\leq \omega_E$ $E(T) = 3N \cdot \frac{\hbar\omega}{e^{\hbar\omega/T} - 1}$.



$\Rightarrow C(T) = C_D(T, \omega_D) + C_E(T) \approx \frac{3}{2}N + \frac{3}{2}N$

2/13/2017.

e.g. ~~4~~ 4. Maxwell distribution (ideal gas, no inner freedom).

ass(1). $\langle N \rangle \ll 1$ (Boltzman limit).

ass(2) $H_{1-p} = \frac{p^2}{2m} + V$. $d\Gamma = \frac{d^3x\, d^3p}{(2\pi\hbar)^3}$

ass(0) $dw = C_{1-p} \exp[-H/T] \cdot d\Gamma$

case (1). $u = V(x) \equiv 0$.

$$1/C_{1\text{-}p} = \int e^{-\frac{p^2}{2mT}} \cdot d\Gamma = V \cdot \left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} = Z_{1\text{-}p}.$$

$\langle E \rangle = \frac{3}{2}T$ and $\langle V_x^2 \rangle = 3T/m$.

case (2). $u \neq 0$ and think of several particles.

☆ $d\Gamma = \frac{1}{N!} \prod d\Gamma_i$ or $\cancel{\prod d\Gamma_i}$ $\prod_{x_i \in x_{\text{the}\cdots}} d\Gamma_i$ (interesting).

case (2.1). $dw_\Gamma = C_N d\Gamma_\Gamma \cdot e^{-\Sigma \frac{1}{T} H}$

$$\Rightarrow Z_N = C_N^{-1} = \frac{1}{N!} Z_{1\text{-}p}^N \Rightarrow F_N = TN\,[\underbrace{\lg N - 1}_{\text{Boltzmann}} - \underbrace{\lg Z_{1\text{-}p}}_{\text{single partition}}]$$

it's easy to check $F_{N+N} \neq 2F_N$!!

$$\Rightarrow S = -(\partial F/\partial T)_V = N \cdot [\ln(V/N) + \tfrac{3}{2}\ln(\tfrac{mT}{2\pi\hbar^2}) + \tfrac{5}{2}]$$

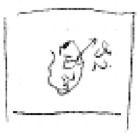
and for no Boltzmann counting,

case (2.2). $S_{\text{no-distinguish}} = N \cdot [\lg V + \frac{3}{2}\ln(\frac{mT}{2\pi\hbar^2}) + 1]$

$S_{1+2} > S_1 + S_2$. (~~true~~ true for gas 1 $\neq$ gas 2).

2/15/2017

16 ~~micro~~ grand-canonical distribution



ass 0. $W_{E,N} = \Delta\Gamma = e^{S_{\text{env}}(E_{\text{env}}, N_{\text{env}})}$ 　 $\begin{cases} E_{\text{env}} = E_{\text{total}} - E_1 \\ N_{\text{env}} = N_{\text{total}} - N_1. \end{cases}$

$$\Rightarrow W_{E,N} = C \cdot e^{-E/T + \mu N/T}$$

$$\Rightarrow Z_G = \sum_{E,N} e^{-E/T + \mu N/T} \quad (N! \text{ is in } e^{\mu N/T}).$$

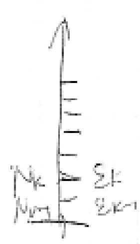
question: what's the main physical difference between this discussion and the previous discussion on many particles?

$$S = \lg Z_G + \frac{\langle E \rangle}{T} - \mu\frac{\langle N \rangle}{T}.$$

✓ $\Omega = F - \mu\langle N \rangle = -T\ln Z_G$. (grand potential).

$W = \exp[(\Omega + \mu N - E)/T]$ 　 μ is variable here !!!!!

e.g. 1. $|m\rangle = |N_1 \cdots N_L\rangle$.



$$Z_G = \prod_k \left(\Sigma_{N_k} (e^{\frac{\mu - \varepsilon_k}{T}})^{N_k}\right). \quad \begin{cases} \text{fermi} & N_k = 0, 1. \\ \text{Boson} & N_k = 0, 1, 2 \cdots \end{cases}$$

✓ $N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{E,V}$.

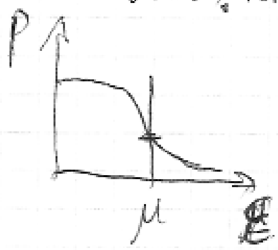
remark: $\mu = \mu_{\text{env}}$ (static potential), it could be changed !!!!!

$1 + 2 = 3$ 　 thus photon won't have this kind of thing
$\mu_1 + \mu_2 = \mu_3$ 　 nor is other non-interactive particles

the meaning of μ is not ~~trivial~~ observable in Hamiltonian !!!

12.

ass(1) $N_k = -\frac{\partial \Omega^k}{\partial \mu} = \frac{e^{(\mu-\varepsilon_k)/T}}{1+e^{(\mu-\varepsilon_k)/T}}$ Fermi - Dirac distribution.

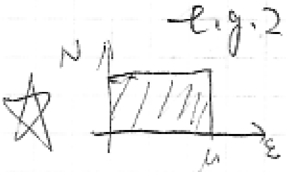


$\begin{cases} \varepsilon_k \gg \mu & e^{(\mu-\varepsilon_k)/T} \\ \varepsilon_k \ll \mu & 1 \end{cases}$ Boltzmann. limit

2/17/2017.

~~another condition~~

e.g.2. as $T \to 0$. $N_k = \begin{cases} 1 & \varepsilon < \mu \\ 0 & \varepsilon > \mu \end{cases}$

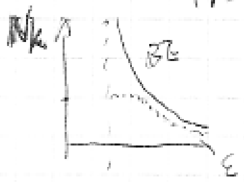


$N = \int_0^\mu d\varepsilon \cdot \frac{dN}{d\varepsilon} = V \cdot g \int_0^{k_F} \frac{4\pi^2 k dk}{(2\pi)^3} = V \cdot g \cdot \frac{k_F^3}{6\pi^2}$ $\quad (k_F^2 = \frac{2M}{\hbar^2}\mu)$.

apply this to metals:

$k_F = (3\pi^2 n)^{1/3}$, $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$

ass (1') Bose - Einstein distribution.



$Z_k = \frac{1}{1-e^{(\mu-\varepsilon_k)/T}}$ $\quad N_k = \frac{1}{e^{(\varepsilon_k-\mu)/T}-1}$ $\begin{cases} \varepsilon-\mu \gg T. & e^{-(\varepsilon_k - \mu)/T} \text{ const} \\ 0 < \varepsilon-\mu \ll T. & \frac{T}{\varepsilon_k - \mu}. \end{cases}$

thus for Bose - Einstein distribution / Fermi Dirac distribution's limit to $\varepsilon-\mu \gg T$ we know that they goes back to the former statistic system

discussion 1: Fermi level gas $0 \le n_k \le 1$.

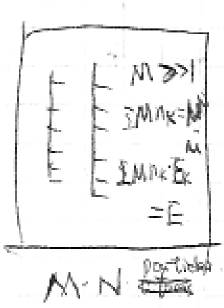
$S_k = -n_k \ln n_k - (1-n_k)\ln(1-n_k)$.

$n_k^{FD} = \frac{1}{e^{(\varepsilon_k-\mu)/T}+1}$

$S = \Sigma\, S_k$ under $N = \text{const}$ + $E = \Sigma\, \varepsilon_k n_k = \text{const}$. !!!

this is indeed a detailed discussion on the micro-interpretation of our variables and by doing this, we got distribution!!!

discussion 2: make copy for our system (for Bosons right now).



totally $M \cdot N$ bosons., [see here we see the relation between μ and N!!!]

$\boxed{M \cdot n_k \text{ bosons} \longleftrightarrow M \text{ brakets.}}$ $\Rightarrow \Delta\Gamma = C^{M}_{Mn_k+M}$

$S_M^{(k)} = \ln \Delta\Gamma = M \cdot [(1+n_k)\ln(1+n_k) - n_k \ln n_k]$

$S^{(k)} = \frac{S_M^{(k)}}{M}$ maximize it. all done. (same result as our partition function ...)

02/20/周 2017.

Summary for both discussion: (1) (monoatomic) $dN = \frac{d^3x\, d^3p}{(2\pi\hbar)^3} e^{(\mu-\varepsilon)/T}$

$$dN' = V \cdot \frac{d^3p}{(2\pi\hbar)^3} \cdot e^{(\mu - p^2/2m)/T} \quad (\text{for } V(x)=0).$$

and partition function:

$$Z_N = \frac{1}{N!} Z_{1\text{-}p}^N \rightarrow F = NT\left[\ln N/V - 1 - \tfrac{3}{2}\ln T - \underbrace{\tfrac{3}{2}\ln \tfrac{m}{2\pi\hbar^2}}_{\text{translational motion}}\right]$$

$$\Rightarrow C_{vm} = \tfrac{3}{2}. \Rightarrow P = NT/V.$$

$$\Rightarrow G = F + PV = NT\left[\ln N/V - \tfrac{3}{2}\ln T - \tfrac{3}{2}\ln\tfrac{m}{2\pi\hbar^2}\right]$$

(2). $Z_{1\text{-}p} \xlongequal{\text{interaction}} V \cdot \int \frac{d^3p}{(2\pi\hbar)^3} \sum e^{-\frac{1}{T}(p^2/2m + \varepsilon_k^{int})} = Z_{1\text{-}p}^{free} \cdot \underbrace{\sum e^{-\varepsilon_k^{int}/T}}_{} \triangleq Z^{int}$

$$F \xlongequal{\text{interaction}} NT\left[\ln\tfrac{N}{V} - 1 - \tfrac{3}{2}\ln T - \tfrac{3}{2}\ln\tfrac{m}{2\pi\hbar^2} - \ln Z^{int}\right]$$

$$G \xlongequal{\text{interaction}} NT\left[\ln P - \tfrac{5}{2}\ln T - \tfrac{3}{2}\ln\tfrac{m}{2\pi\hbar^2} - \ln Z^{int}\right]$$

1. diatomic gas o——o. $E = E_{tran} + E_{vir} + E_{rot}$

① K = angular momentum $k = 0, 1, \dots$ $\hat{k}^2 = k\cdot(k+1)$

$$\varepsilon_k = \frac{\hbar^2}{2I}\cdot k(k+1) \qquad Z_{rot} = \sum e^{-\varepsilon_k/T}\cdot(2k+1).$$

case(1) $\hbar^2/2I \gg T$. $Z_{rot} \rightarrow 1 + 3e^{-\hbar^2/IT}$

$$F_{rot} = -NT\ln Z_{rot} \simeq -3NT\, e^{-\hbar^2/IT}$$

$$E_{rot} = 3N T^2 e^{-\hbar^2/IT}\cdot\left(\frac{\hbar^2}{IT^2}\right) = 3N\frac{\hbar^2}{I}\cdot e^{-\hbar^2/IT}$$

Case (2). $T \gg \hbar^2/2I$

$$Z_{rot} \simeq \int_0^{+\infty} 2k\, dk\, e^{-\hbar^2k^2/2IT} = \frac{2IT}{\hbar^2}$$

$$F_{rot} \simeq -NT\left[\ln T + \ln\frac{2I}{\hbar^2}\right] \quad (\text{compare this with } F_{translational}).$$

☆ remark: notice that we make an approximation that is.

$\sum_k \rightarrow \int dk$, this should be modified because for different kinds of symmetry, we will have some modification from the spin.

i.e. $S=0$ even $0, 2, \dots$ $\times 1$.

$S=1$. odd $1, 3, \dots$ $\times 3$

② Vibration. (non-symmetric)

?

$$Z_{vir} = \sum e^{-\hbar\omega n/T} = 1/(1 - e^{-\hbar\omega/T})$$

$$\Rightarrow F \simeq -NTe^{-\hbar\omega/T}$$

thus we total expression for diatomic gas. like F, μ. balabala.

e.g. then for chemical process, $G = \sum \mu_i N_i \longrightarrow$ should be minimal

$\Rightarrow \Delta G_1 - G_2 = 0$ or $G = \text{const}$

14. 02/22/2017

e.g. H-plasma. $\begin{cases} H^{\ddagger} = p+e \\ p+e = H \end{cases}$ $\rightarrow$ equilibrium at T.

ass(1) $\Rightarrow$ $\mu_e + \mu_p = \mu_H - E_I \rightarrow 13.6\text{eV}$. (weakly ionized plasma).

monoatomic gas $\begin{cases} g_e = g_p = 2, \quad g_H = 4 \\ \mu_x = T \cdot [\ln n_x - \frac{3}{2}\ln T - \frac{3}{2}\ln\frac{m_x}{2\pi\hbar^2} - \ln g] \end{cases}$

$$\Rightarrow \ln[\frac{n_e n_p}{n_H}] = \frac{3}{2}\ln T + \frac{3}{2}\ln[\frac{m_e}{2\pi\hbar^2}] - E_I/T$$

$$\Leftrightarrow \quad n_H = n_e n_p \cdot (\frac{2\pi\hbar^2}{m_e T})^{3/2} \cdot e^{+E_I/T}$$

ass(2). Saha's equation $\begin{cases} n_e = n_p \\ n = n_e = n_H \end{cases}$

18. quantum statistics (~~ideal~~ ideal gas, no other freedom)

discussion (classical). : (1) occupancy number $n = e^{(\mu-\varepsilon)/T} \ll 1$. per $\frac{d^3x\, d^3p}{(2\pi\hbar)^3}$ (i.e. per state)

(2). $e^{\mu/T} \ll 1 \Rightarrow e^{\mu/T} = \frac{N}{gV} \cdot (\frac{2\pi\hbar^2}{mT})^{3/2} \overset{\Delta}{=\!=} (2\pi\, T_0/T)^{3/2}$

$V/N \sim r^3 \Rightarrow T_0 = \frac{\hbar^2}{m} g^{-2/3} r^{-2}$

(3). $T \sim T_0 \Rightarrow \lambda_T = \frac{\hbar}{p} = \frac{\hbar}{\sqrt{mT}} \gtrsim r \Leftrightarrow T \lesssim T_0$

then we must use quantum statistics.

e.g. $n_{F/B} = 1/[e^{(\varepsilon-\mu)/T} \pm 1]$

$$N_{F/B} = \int d\omega \cdot n_{F/B} = \int (\frac{dx\,dp}{2\pi\hbar})^3 \cdot 1/[e^{(\varepsilon-\mu)/T} \pm 1]$$

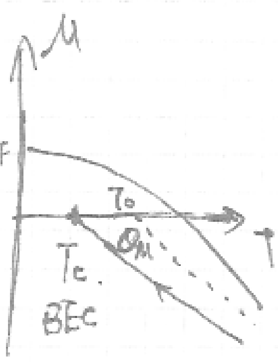
$$= \frac{g \cdot V \cdot m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^{+\infty} \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{(\varepsilon-\mu)/T} \pm 1} \qquad (z \overset{\Delta}{=} \varepsilon/T).$$

$$= \underbrace{V \cdot T^{3/2} \cdot \frac{g\, m^{3/2}}{\sqrt{2}\pi^2\hbar^3}}_{C} \int_0^{\infty} \frac{\sqrt{z}\, dz}{e^{z-\mu/T} \pm 1}$$

Case (1). $\overset{e^{\mu/T} \ll 1}{=\!=}$ $C \cdot e^{\mu/T} \int_0^{+\infty} \sqrt{z}\, dz\, e^{-z} [1 \mp e^{\mu/T} e^{-z}]$

$$= C \cdot e^{\mu/T} [\Gamma(\tfrac{3}{2}) \mp 2^{-3/2} \Gamma(3/2) e^{\mu/T}] \quad (\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{\sqrt{\pi}}{2}).$$

$$= C \cdot e^{\mu/T} \frac{\sqrt{\pi}}{2} \cdot [1 \mp 2^{-3/2} e^{\mu/T}]$$



i.e. $\frac{N_{F/B}}{g \cdot V} \cdot (\frac{2\pi\hbar^2}{mT})^{3/2} = e^{\mu/T} [1 \mp 2^{-3/2} e^{\mu/T}]$

which is quite different from the previous discussion.

$\Rightarrow \mu_{classical} \approx \mu \mp T \cdot 2^{-3/2} e^{\mu/T}$

$\boxed{\text{remark: rework of } \mu \text{ is important. like in E.M., } J = \sigma E. \quad \nabla\times B = \mu_0 J, \quad \nabla\times E = -\partial_t B, \text{ it's self-consistent}}$

this page is for compensation and summary

14.5.

$$\mathcal{E} = \int_0^{\infty} d\varepsilon\, \varepsilon \frac{g(\varepsilon)}{\cancel{d\varepsilon}} \cdot \frac{1}{e^{(\varepsilon-\mu)/T}+1} = \frac{gV\cdot m^{3/2}}{\sqrt{2}\pi\hbar^3}\int_0^{+\infty} \frac{\varepsilon^{3/2}\, d\varepsilon}{e^{(\varepsilon-\mu)/T}+1}.$$

$$= V\cdot T^{5/2}\cdot \frac{gm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int \frac{z^{3/2}dz}{e^{z-\mu/T}+1}.$$

$$F = \mp T\int_0^{+\infty} d\varepsilon \cdot g(\varepsilon)\cdot \ln[1\pm e^{(\mu-\varepsilon)/T}]$$ (good point!)

$$= \mp VT^{5/2}\frac{gm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^{+\infty}\sqrt{z}\,dz\cdot \ln[1\pm e^{\mu/T-z}]$$

$$= -\frac{2}{3}E. \Rightarrow p = -\frac{\partial F}{\partial V} = -\frac{F}{V} = \frac{2}{3}E/V.$$

remark: ① adiabatic process have: $dE = TdS - pdV = \frac{3}{2}pdV + \frac{3}{2}Vdp = -pdV$

$\Longrightarrow p^3V^5 = \text{const} \Leftrightarrow pV^{5/3}$. (classical ~~gas~~ ideal gas).

② $\gamma = C_p/C_V \Rightarrow pV^{\gamma} = \text{const}$.

continue: $e^{\mu/T} \ll 1$

$$E \approx \frac{3}{2}V\cdot T^{5/2}\frac{gm^{3/2}}{(2\pi\hbar)^{3/2}}\cdot e^{\mu/T}[1\mp 2^{-5/2}e^{\mu/T}].$$

$$P = T^{5/2}\frac{gm^{3/2}}{\hbar^{3/2}}\, e^{\mu/T}[1\mp 2^{-5/2}e^{\mu/T}].$$

$$\Rightarrow \frac{PV}{NT} = \frac{1\mp 2^{-5/2}e^{\mu/T}}{1\mp 2^{-3/2}e^{\mu/T}} = 1\pm 2^{-5/2}e^{\mu/T} = 1\pm 2^{-5/2}\cdot\frac{N}{gV}\cdot\left(\frac{2\pi\hbar}{mT}\right)^{3/2}$$

$$= 1\pm 2^{-5/2}\left(2\pi\cdot\frac{T_0}{T}\right)^{3/2}$$

Compare P and P_{class} we could say $P > P_{class}$ is repulsion, otherwise attraction

for BEC, the lowest order needs to be treated seperately, e.g.

$N \gg 1$.

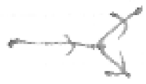
the lowest order is const rotation which would not be treated as H.O.

and that part has $\langle E\rangle = \langle \frac{T}{2}\rangle$

03 / 29 / 2017

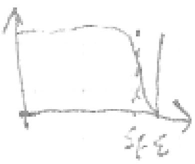
discussion: ① $n^F(\varepsilon) = 1/[e^{(\varepsilon-\mu)/T}+1]$

think of $T \ll \varepsilon-\mu$. $T \to 0$ $\mu \to \varepsilon_F$



scattering? $\sigma \sim |\langle f|V_{int}|i\rangle|^2 \, \tau_{final}$

$\boxed{\text{occupied thus not frequently happen} \to \text{speed survive longer}}$



in formal gas

② $N^F(\varepsilon) = \int d\varepsilon\, n(\varepsilon)\, g(\varepsilon) \overset{T=0}{=} \int_0^{\varepsilon_F} d\varepsilon \dfrac{gV \cdot m^{3/2}\sqrt{\varepsilon}}{\sqrt{2}\pi^2 \hbar^3}$

$= gV \cdot \dfrac{\sqrt{2}\, m^{3/2} \varepsilon_F^{3/2}}{3\pi^2 \hbar^3}$

$\Rightarrow p_F = \sqrt{2m\varepsilon_F} = (6\pi^2 \frac{N^F}{gV})^{1/3} \cdot \hbar$

③. $g(\varepsilon) = g(\varepsilon_F) \cdot (\varepsilon/\varepsilon_F)^{1/2}$. $g(\varepsilon_F) = \frac{3}{2} N^F/\varepsilon_F$

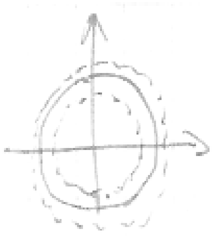
$E = \int_0^{\varepsilon_F} d\varepsilon \cdot \varepsilon\, g(\varepsilon) = \frac{3}{5} N^F \cdot \varepsilon_F = \frac{2}{5} g(\varepsilon_F) \cdot \varepsilon_F^2$

☆ $p^F \cdot V^{5/3} = \text{const} \Rightarrow (p^F V = \frac{2}{3}E)$

$T=0$ compression is also adiabatic $S = \text{const} = 0$.

$\boxed{\text{Condition: add a new particle into the system then compress it}}$.

④. $I_\varphi = \int_0^{\infty} d\varepsilon\, \varphi(\varepsilon)\, n(\varepsilon) = + \int_0^{+\infty} d\varepsilon\, \Phi(\varepsilon) (-\frac{\partial n}{\partial \varepsilon})$



now we expand $\Phi(\varepsilon)$ around $\varepsilon = \mu$. (according to $-\frac{\partial n}{\partial \varepsilon}$)

$I_\varphi \approx \Phi(\mu) + \frac{T^2\pi^2}{6}\varphi'(\mu)$ $\longrightarrow$ a kind of "effective width".

e.g. 1 $N^F = \int_0^\infty n(\varepsilon) g(\varepsilon)\, d\varepsilon = \cancel{N^F}\ \underbrace{\int_0^\mu g(\varepsilon)\, d\varepsilon} + (\cancel{\quad}) g'(\mu) \frac{\pi^2 T^2}{6}$

$\approx N^F + \int_{\varepsilon_F}^{\mu} (\mu - \varepsilon_F) g(\mu)$

$\Rightarrow \mu = \varepsilon_F - \frac{1}{6}\pi^2 T^2\, g'(\varepsilon)/g(\varepsilon)$.

$\boxed{\text{idea here is: calculate } T=0 \text{ then go back to } T \neq 0 \text{ to get finite results, i.e. perturbation.}}$

☆ e.g. 2. $E^F = \ldots\ldots$

$\Rightarrow E^F(T\neq0) - E^F(T=0) = \frac{\pi^2 T^2}{6} g(\mu) \Rightarrow \cancel{C_V \propto TN}$

always like $\varphi(\varepsilon) = g(\varepsilon) \cdot \alpha(\varepsilon)$ $\quad C_V = \frac{\pi^2}{2} \cdot \frac{T}{\varepsilon_F} N^F$

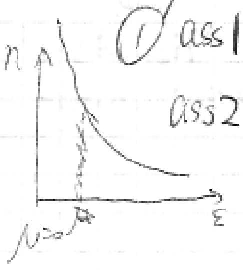
typically $\varepsilon_F \sim$ few eV $\sim$ few 10^4 K.

16. 03/03/2017.

⑤ $n_F = 1/(e^{(\varepsilon-\mu)/T}+1)$ $\quad 1-n_F = 1/(e^{(\mu-\varepsilon)/T}+1)$

these parts are the fermi holes. e.g. semiconductor bands

17. Boson degenerate gas.



① ass1. $n^B(\varepsilon) = 1/(e^{(\varepsilon-\mu)/T}-1)$ $\quad \varepsilon \geq 0 \Rightarrow \mu \to 0.$

ass2. $N^B_{\mu=0} = \underbrace{\int_0^\infty d\varepsilon\, g(\varepsilon) \frac{1}{e^{\varepsilon/T}-1}}_{N^B(T,\mu=0)} + N_{BEC}$ ← ground state numbers

$N^B = \int_0^\infty d\varepsilon \cdot \frac{gVm^{3/2}\sqrt{\varepsilon}}{\sqrt{2}\pi^2\hbar^3} \cdot \frac{1}{e^{\varepsilon/T}-1} = gV \frac{m^{3/2}\cdot T^{3/2}}{\sqrt{2}\pi^2\hbar^3} \underbrace{\int_0^\infty \frac{dz\sqrt{z}}{e^z-1}}$ $\left(=\zeta(\tfrac{3}{2})\cdot\Gamma(\tfrac{3}{2})\right)$

☆ ass3 $T_0 \triangleq \frac{\hbar^2}{m}\left(\frac{N}{gV}\right)^{3/2} \Rightarrow \left(2\pi\frac{T_0}{T}\right)^{3/2} = \frac{2}{\sqrt{\pi}}\zeta(\tfrac{3}{2})\Gamma(\tfrac{3}{2})$

critical point $\Rightarrow N_{total} = N^B(\mu=0, T_c)$.

$T_c = 2\pi T_0 \left[\frac{2}{\sqrt{\pi}}\Gamma(3/2)\zeta(3/2)\right]^{2/3} \approx 3.133 T_0$ (temperature BEC quantum effects)

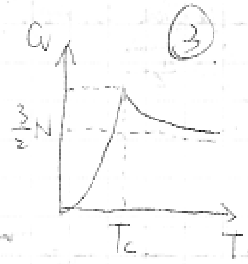
$\Rightarrow N^{BEC}_{T<T_c} = N_{Total}\left(1-(T/T_c)^{3/2}\right)$ ✓

② $E^B_{\mu=0}(T<T_c) = \int_0^\infty d\varepsilon\, \varepsilon\, g(\varepsilon) \frac{1}{e^{\varepsilon/T}-1} \sim T^{5/2}$

$E(T_c) = N\cdot T\cdot\left(\frac{3}{2}\cdot\frac{\zeta(5/2)}{\zeta(3/2)}\right)\cdot N = 0.7701\, N\cdot T_c$ ~~$\sim T^{5/2}$~~

ass4 $PV = \frac{2}{3}E \Rightarrow P_{(T_c)} = \frac{2}{3}\frac{E}{V} = 0.5134\cdot\frac{NT_c}{V}$

$\Rightarrow P(T<T_c) = P(T_c)\cdot(T/T_c)^{5/2}$ (no longer with V, easy to verify.)



③ $C_V(T<T_c) = \frac{5}{2}E/T$

$C_V(T_c) = 1.925N$ $\quad (> 3/2\, N = C_V(\text{ideal gas}))$

$He^4 \to 2.18\,K$ experiment

$\hookrightarrow 3.14\,K$. T_c BEC estimation.

④ $S = 0 \quad (T=0)$.

$S = \int_0^T dT' \frac{C_V(T'<T_c)}{T'} = \frac{5}{3}\frac{E(T<T_c)}{T}$

(Bose or Fermi) $F = -\frac{2}{3}E$ $\quad G = 0 = \mu N \Rightarrow \mu = 0$ again (a must).

⑤ $P_{saturated} = P(T)$. from $G=0$.

$SdT = VdP \Rightarrow \frac{dP}{dT} = \frac{S}{V} = \frac{S_m}{V_m} \xlongequal{\text{CC theorem}} \frac{\Delta S_m}{\Delta V_m} = \frac{\Delta Q}{T(V_1-V_2)}$

$\Rightarrow V^{BEC} \equiv 0$!!! (first order ?)

What is confusing is that $p=0$. (second order !?)

※ notice that $\int d^n p \frac{1}{e^{p^2/2m-\mu}-1}$ is divergent for $n=2$!?

03/06/2017

⑥ with a potential. e.g. $V = \cancel{\frac{1}{2}m\omega}\ \frac{1}{2}kx^2$

$N \propto T^3/T_c^3$ rather than $(T/T_c)^{3/2}$ anymore.

ass 1 $\quad n = \frac{N}{V} \ll \lambda^{-3} = \left(\frac{2\pi\hbar^2}{mT}\right)^{-3/2}$ thermal wave length.

ass 2. hard core model, and neglect 3 or more body interaction (or just make a mean field). then:

$$E_N = \sum \frac{p_i^2}{2m} + \sum_{i<j} u_{12}(\vec{x}_i - \vec{x}_j).$$

$$\Rightarrow Z = \frac{V^N}{N!} \int \prod \frac{d^3p}{h^3} e^{-p^2/2mT} \cdot \underbrace{\frac{1}{V^N}\int e^{-U/T}}$$

approximation 1.

$$\frac{1}{V^N} \cdot \frac{N(N-1)}{2} \cdot \int dx_1 dx_2\, e^{-u_{12}(x_1-x_2)} \cdot V^{N-2}$$

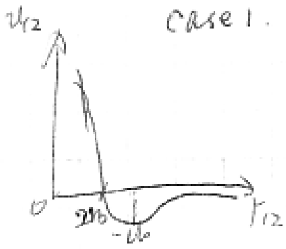
$$= 1 + \frac{N^2}{2V}\int d^3x\left(e^{-u_{12}(x)/T} - 1\right) \triangleq 1 - \frac{N^2}{V}B(T).$$

$$\Rightarrow F_N \approx F_N^{ideal} + T\cdot\frac{N^2}{V}\cdot B(T).$$

$$P \approx -\left(\frac{\partial F}{\partial V}\right)_T = P^{ideal} + - T^2 \cdot \frac{1}{V_m^2}\cdot B(T)$$

$$G = G^{ideal} + P\cdot N\cdot B(T)$$

$$V = \left(\frac{\partial G}{\partial p}\right)_T = V^{ideal} + N\cdot B(T)$$



case 1.

$B(T) \longrightarrow T \ll u_0 \quad \Rightarrow \int (1 - e^{u_0/T})\, dx^3 \ll 0.$

$\quad\ \longrightarrow T \gg u_0 \quad \Rightarrow \int_0^{2r_0} + \int_{2r_0}^{\infty} \Rightarrow \frac{1}{2}\cdot\frac{4\pi}{3}(2r_0)^3 + \frac{1}{2}\int \frac{u(x)}{T} dx^3$

$$\Rightarrow \left(\frac{\partial T}{\partial p}\right)_H = \frac{1}{C_p}\left[T\left(\frac{\partial V}{\partial T}\right)_p - 1\right] = \frac{N}{C_p}\cdot\left(T\cdot B'(T) - B\right).$$

$$\begin{cases} T < T_{int} & \left(\frac{\partial T}{\partial p}\right)_H > 0 \Rightarrow \Delta T < 0. \\ T > T_{int} & \left(\frac{\partial T}{\partial p}\right)_H < 0. \Rightarrow \Delta T > 0. \end{cases} \quad (\Delta p < 0)$$

$\Rightarrow$ Van de Waals equation

$$p + \frac{a}{V_m^2} = NT\left[\frac{1}{V} + \frac{Nb}{V^2}\right] \qquad \begin{cases} a = \frac{1}{2}\int_0^{2r} dx^3 \\ b = \frac{1}{3}\int_{2r_0}^{\infty} \frac{u(r)}{T} u(r)\, dx^3 \end{cases}$$

20. Cluster expansion (expansion in density)

$$Z_G = \sum_N Z_N e^{\mu N/T}$$

$E_0 = 0 \qquad\qquad Z_0 = 1$

$E_1 = \frac{p^2}{2m} \qquad\quad Z_1 = \int d\Gamma_1 e^{-E/T} \left(= V\cdot\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2}\right)$

$$E_n = \sum \frac{p_i^2}{2m} + u_{2\cdots n} \quad Z_n = \frac{1}{n!}\int \prod d\Gamma_i\, e^{-E_n/T} = \frac{1}{n!}(Z_{1\text{-}p})^n \cdot \underbrace{\frac{1}{V^{n-1}}\int dx_{12}^3\, dx_{13}^3\, dx_{14}^3 \cdots e^{-u_{12\cdots n}/T}}_{\triangleq I_n}$$

18.

$$Z_G = \sum \frac{I_N}{N!}\left(Z_{1\text{-}p}\, e^{\mu/T}\right)^N$$

$\Omega = -T \ln Z_G \quad \Rightarrow \quad P = \frac{T}{V}\ln Z_G$ ← ideal gas, not so general? (for I_N don't give us anything we want)

Case 1. $u \equiv 0. \Rightarrow p \overset{\text{ideal}}{=} \frac{T}{V}\cdot Z_{1\text{-}p}\, e^{\mu/T} (=N)$

Case 2. non-zero u :

$$p = \frac{T}{V}\ln Z_G = \frac{T}{V}\sum \frac{J_N (Z_{1p}\, e^{\mu/T})^N}{N!}$$

$\Rightarrow J_1 = I_1 = 1$.

$J_2 = I_2 - 1 = \frac{1}{V}\int dx^3 \left(e^{-u/T} - 1\right) = -2B(T)\cdot V$ (2nd Virial coefficient).

$J_3 = I_3 - 1 - 3(I_2 - 1) \propto \left(\,{}^{1}{}_{3}{}^{2}\right) - \sum_{\text{sym}}\left(\,{}^{1}{}_{3}{}^{2}\right)$

$$\Rightarrow p = T\cdot\left[\frac{N}{V} + \left(\frac{N}{V}\right)^2 B(T) + \left(\frac{N}{V}\right)^3 C(T) + \cdots\cdots\right].$$

3/10/2017.

21. phase transition : macro ~~para~~ parameters suddenly change

① order of transition : (1) 1st order → disnuity in density ~~ ~~ $V_m = V/N$

entropy $S_m \neq = S/N$

notice that $-\left(\frac{\partial G}{\partial T}\right)_p = S$. $V = \left(\frac{\partial G}{\partial p}\right)_T$

idea is : $\mu_1(p,T) = \mu_2(p,T)$. $d\mu_1 = d\mu_2$ could gives us the equation

$$\frac{dp}{dT} = \frac{\Delta S}{\Delta V} = \frac{S_2 - S_1}{V_2 - V_1} = \frac{L}{T\cdot \Delta V}$$

$\boxed{\text{remark: experimentally we could somehow use this, because } L \text{ we measure, so is } \Delta V \text{ and we know } T \text{ then it seems to be clear.}}$

(2). second order : $\frac{\partial}{\partial T} C_V = \left(\frac{\partial^2 E}{\partial^2 T^2}\right)_V$ ← thermodynamic limit $\begin{cases} V \to \infty \\ N \to \infty \\ V_m \text{ fixed}\end{cases}$

(3). ~~Cross over~~ 3rd order : cross over

② density correlation : $\langle \delta\rho(x)\, \delta\rho(y)\rangle \propto e^{-|x-y|/\varepsilon}$ → correlation length.
(1st order)

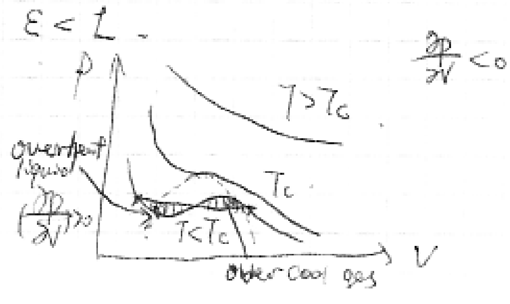
$$\varepsilon \xrightarrow[T \to T_c]{} \infty$$

~~#~~ let $V = L^3$ then $\varepsilon < L$.

example 1. Van de Waals

$$p + \frac{a}{V^2} = \frac{T}{V-b}$$

Competition between push and ~~ ~~ expand

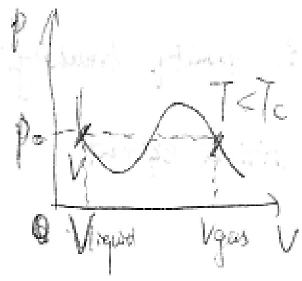


③ mixture of phases : $\begin{cases} \mu_1 = \mu_2 & \text{chemical } (E, S) \\ T_1 = T_2 & \text{thermo} \\ p_1 = p_2 & \text{mechanical} \end{cases}$

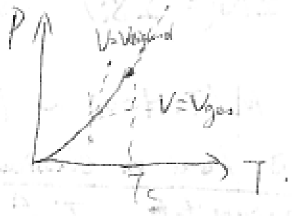
$G_1 = G_2 \Rightarrow \int_1^2 dG = 0 = \int_1^2 V dp \Rightarrow$ gives you p_0.

i.e. I find the p_0 related to T.

$T_C = \frac{8a}{27 \cdot b}$ $p_C = \frac{a}{27 b^2}$ $V_C = 3b$



$\Rightarrow$



where is μ ?
isn't that a uniform change?
reconstruct itself ?

④ example of water triple point

$\mu_{water}(T, p) = \mu_{ice}$ $\mu_{water} = \mu_{vapor}$. always same T, p.

$\Rightarrow T_{tri} = 273.16K$. $p_{tri} = 0.6\ kPa$

generally speaking :

mixture of $A, B \cdots$ material

$T, P, C_{1,2\cdots}^{A,B\cdots}$ can exist $1, 2 \cdots R$ phases

$\mu_r^A(P, T) = \mu^A$ gives you $R-1$ equations (not R)

and we have L species of materials

$\Rightarrow$ totally $2 + R(L-1)$ parameters.

$L \cdot (R-1)$ equations : $n = 2 + R(L-1) - L(R-1) =$ freedom ≥ 0

3/20/2017.

⑤ classification of transitions :

1st : $\Delta S_{12} \neq 0$. $\ell = T \cdot \Delta S$

2nd : $\begin{cases} \Delta S = 0 \\ \Delta V = 0. \end{cases}$ $V_m = \frac{1}{N}\left[\frac{\partial G}{\partial p}\right]_T$

$\begin{cases} \text{feromagnetic} & \uparrow\uparrow\uparrow \quad T<T_c \\ \text{antiferomagnetic} & \uparrow\downarrow\uparrow \end{cases}$

remark: $\lim (N \to \infty)$
and limit ($\to 0$)
is important in who is first!

e.g. 1. $\begin{cases} \text{ferromagnetic} \\ \text{antiferromagnetic} \end{cases}$

e.g. 2. Bose - Einstein condensation.

$\psi_{BEC}(x) = r(x)\, e^{i\varphi}$ order parameter

degenerate global gauge symmetry

$\boxed{\text{Hamiltonian is } \sout{\text{no longer conserved}} \text{ conserved under this symmetry, but not these ground states}}$

20.

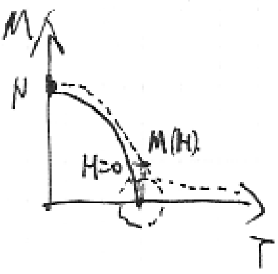
22. magnitization : $H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - H \sum \sigma_i$ $\begin{cases} J>0 & \text{fero.} \\ J<0. & \text{antifero} \end{cases}$

① Ising model : classical spin.

ass 1. $\mathcal{H} = -J \sum S_i S_j - H \sum S_i$ $S_i = \pm 1$. Z_2 degeneracy

ass 2. $\Rightarrow$. ground state $\Rightarrow$ $\mathcal{H} = -J \cdot N \cdot \dim - H \cdot N$.

ass 3. $T < T_c \Rightarrow M = \pm N (= \sum S_i)$ (spontaneous ~~sy~~ Z_2 symmetry breaking ?)



Conclusion : $T < T_c$, $M(T, H=0) \sim |T - T_c|^{\beta}$ ← critical exponent.

derivation is based on classical cases.

$\boxed{\text{Temperature scaling parameter} \quad \tau \triangleq \frac{T - T_c}{T_c}}$

$C_H = -\frac{T}{N} \frac{\partial^2 G}{\partial T^2} \propto |\tau|^{-\alpha}$ $M_m = \frac{M}{N} = \begin{cases} \tau^{\beta} & \tau < 0 \\ 0 & \tau > 0 \end{cases}$

Gibbs potential :

$Z(T, H) = \sum e^{-\frac{1}{T}(H_0 - H_n)} = e^{-\frac{G(T,H)}{T}}$

$\chi = \left(\frac{\partial m}{\partial H}\right)_T = -\frac{1}{V}\left(\frac{\partial^2 G}{\partial H^2}\right)_T \propto |\tau|^{-\gamma}$

Case 1. $\tau = 0$. $H \neq 0$: $m(T_c, H) \propto H^{1/\delta}$, $H_{eff} \propto m^{\delta}$

correlation function :

$\Gamma(r) = \langle m(r) m(0) \rangle - \langle m(r) \rangle \langle m(0) \rangle$

$\propto |r|^{+2-d-\xi} \exp(-r/\xi)$ (d is dimension)

correlation length

$\xi \propto |\tau|^{-\nu}$ $\nu > 0$

remark : $\alpha, \beta, \gamma, \xi, \nu, \delta$

$T \to 0$: microscopic details become irrelavent "scaling".

the smooth terms after $|\tau|^{\dots}$ would be not important.

because of ~~sta~~ singularity in the begining took most importance.

e.g. $C_H \propto |\tau|^{-\alpha} \propto -\frac{T}{N}\frac{\partial^2 G}{\partial T^2} \Rightarrow \Delta G/V \propto |\tau|^{2-\alpha}$

(1) $H_{eff} \propto \frac{\Delta G}{V} \cdot \frac{1}{m} \propto \frac{M}{\chi}$

$\Rightarrow \alpha + 2\beta + \gamma = 2$

~~Suspe~~ Susceptibility : (2) $\chi = \frac{1}{T}\int dx^3 \Gamma(|x|) \propto \xi^{2-\xi} \quad |\xi|^{2-\xi}$

physical meaning of correlation function :

also. (3) $m \propto |\Gamma(r)|^{1/2} \sim |\xi|^{1-\frac{d+\xi}{2}}$ (4) $H_{eff} \propto |\xi|^{\frac{-2-d+\xi}{2}}$

(5) $\frac{\Delta G}{V} \propto \frac{T_c}{V_{corr}} \propto \xi^{-d} \propto |\tau|^{2-\alpha}$

in fact only four of them are ~~not~~ independent.
thus we get 2 independent exponents

remark:
this 2-independent thing is not coming from Hamiltonian, it could come from some symmetry. i.e. different system, same parameter

$\alpha + 2\beta + \gamma = 2$

$\nu = \frac{2-\alpha}{d}$

$\delta = 1 + \gamma/\beta$

$\xi = 2 - \gamma/\nu$

$\Rightarrow p$ and X (or ξ)

$\Gamma(r) \propto r^{-p} e^{-r/\xi} = \xi^{-p} \underbrace{x^{-p} e^{-x}}$ → universal scale-invariant dimension less. X

all spin correlated

23. Landau mean-field theory (for $\frac{(\delta m)^2}{m^2} << 1$. fluctuation small).

$m(x)$ is magnetic field.

$F(T, M) = V[A(T) \cdot m^2 + B(T) \cdot m^4 - H \cdot m]$ $M = V \cdot m$ $m = \text{const}$

second order phase transition: $A(T_c) = 0$ $T > T_c$

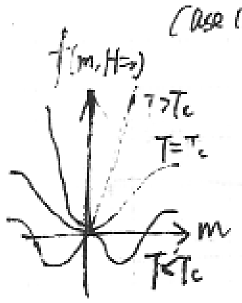
$A(T) = a \cdot \tau$ $T < T_c$

? ①. $F = \int dx^3 f(m, H)$.

$f = -H \cdot m + A(T) m^2 + \frac{1}{2} B(T) \cdot m^4 + C |\nabla m|^2 + O(\cdots)$

Case (1). uniform field: $H = 0$ $\nabla m = 0 \Rightarrow m = \text{const}$.

m is determined by $\min f(m, H=0)$.



$\frac{\partial f}{\partial m} = 2a\tau \cdot m + 2B m^3$

$\Rightarrow \begin{cases} m = 0 & T > T_c \\ m = \pm\sqrt{\frac{a|\tau|}{B}} & T < T_c \end{cases}$ $\Rightarrow \beta = 1/2$

$f(m, H=0) = \begin{cases} 0 & T > T_c \\ -\frac{a^2 \tau^2}{2B} & T < T_c \end{cases}$

$\Rightarrow C = -T_c \frac{\partial^2 f}{\partial T^2} = \begin{cases} T_c^3 \cdot \frac{a^2}{B} & T < T_c \\ 0 & T > T_c \end{cases}$ a jump and $\alpha = 0$.

remark: ① A depends on T ✓ ② B>0. ferromagnet ③. why are they even order, not odd?

answer to ③ because H=0 would give you symmetric terms thus actually we have odd terms, but only think of leading

case (2). $H \neq 0$. $T = T_c$.

$\frac{\partial f}{\partial m} = 0 \Rightarrow m = \left(\frac{H}{2B}\right)^{1/3} \Rightarrow \delta = 3$.

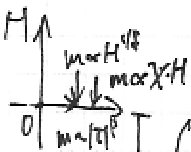
Case (3). $H = 0$, $T \lesssim T_c$, $|\nabla m|$ take into account.

$f = A(T) m^2 + C|\nabla m|^2 \simeq a\tau m^2 + C \cdot \frac{m^2}{\xi^2} \Rightarrow \xi \propto \sqrt{\frac{c}{a\tau}}$

can think of this from Lagrangian equation $\nu = 1/2$

22.

Case (4). $H \neq 0$. $T \lesssim T_c$. uniform.



$f = a\tau m^2 - Hm. \Rightarrow m = \frac{H}{2a\tau}$. $\chi = \frac{\partial m}{\partial H} = \frac{1}{2a\tau} \Rightarrow$

remark: the trick here is compare the order of our terms and only keep those leading two left in the expression to do the calculation. interesting derivation.

also $d = 4$. ~~and~~ but fluctuation dominates gives you $d < 4$.

decorrelate magnetization for $r < \xi$

discussion: $\frac{\langle \delta m^2 \rangle}{m^2} \ll 1 \Rightarrow \frac{T_c}{a|\tau|}\xi^{-d} \cdot \frac{1}{m^2} = \frac{b\, T_c}{a^2|\tau|^2}\left(\frac{a|\tau|}{c}\right)^{d/2} \propto |\tau|^{d/2-2} \ll 1$

$\Rightarrow d > 4$ is a must for MFT.

but for $d < 4$., there might be a region $|\tau| > \tau_0 \Leftrightarrow T < T_c - T_c\tau_0$

e.g. Ginsburg - Landau theory of S.C. (BCS condensate wavefunction)

form (1) $m \to \psi(x)$, $\nabla \to \nabla - i\frac{q}{\hbar}\vec{A}$.

$f \to a\tau|\psi|^2 + \frac{1}{2}b|\psi|^4 + \frac{\hbar^2}{2m}|(\nabla - i\frac{q}{\hbar}\vec{A})\psi|^2 + \frac{B^2}{2\mu_0}$

☆. $\min F = \min \int f\, dV \Rightarrow \frac{\partial F}{\partial \psi^*} = 0 \quad \frac{\partial F}{\partial A} = 0.$

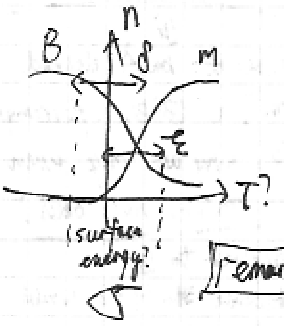
$\Rightarrow \frac{1}{\mu_0}\nabla \times \vec{B} = \frac{i\hbar q}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi + 2i\frac{q}{\hbar}A|\psi|^2) \triangleq \vec{j}_s$ current density?

☆ form(2) $\psi = n_s^{1/2}(x)\, e^{i\varphi(x)}$

$\Rightarrow j_s = \frac{\hbar q}{2m} n_s \cdot [\nabla\varphi - \frac{q}{\hbar}\vec{A}]$

$\Rightarrow \nabla \times \vec{j}_s = -\frac{q^2}{m} n_s \vec{B}$ (London's equation).

$\Delta\vec{B} = \mu_0 \frac{q^2}{m} n_s \vec{B}$ exponential decay !! $\triangleq \delta$.



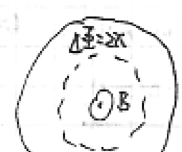
use $n_s = \sqrt{|\psi|} = |m| \propto |\tau|^{1/2} \Rightarrow$ penetrate depth $\propto |\tau|^{-1/2}$

remark: right now two mechanics compete with each other: δ vs. ξ

$\kappa \triangleq \delta/\xi = \text{constant}(\tau)$.

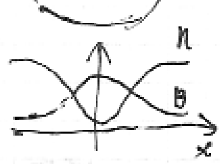
$\kappa < 1/\sqrt{2} \quad \sigma > 0 \quad$ SC I. meissner effect is broken through the entire sample

$\kappa > 1/\sqrt{2} \quad \sigma < 0. \quad$ SC II. vortex exist?



SC current screens the mag field?

~~the~~ remark: the surface energy is from current to B-field. Look up for plasma physics



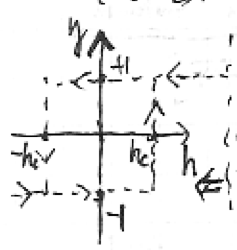
24. Ising model (Mean - field theory)

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} S_i S_j - h\sum S_i \qquad S_i = \pm 1.$$

ass (1). $\langle S_i\rangle = \eta$., $J > 0$. no fluctuation $S = 0$

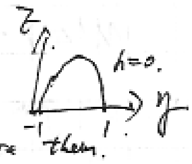
$\Rightarrow F = E = -JNd\cdot\eta^2 - hN\eta$

case (1). $h = 0$. spontaneous magnetization.



$\eta = \pm 1$, min F.

we can somehow draw a diagram



and for all $h \neq 0$. terms, also compare them.

then based on min F is our real case, we could draw another diagram.

where $h_c = 2JN$.

case (2). molecular field theory

ass(2). $S_i = \eta + \tilde{S}_i$, $\langle \tilde{S}_i\rangle = 0$, $\langle \tilde{S}_i \tilde{S}_j\rangle = 0$ $(i \neq j)$ $\qquad \frac{\langle \tilde{S}_i \tilde{S}_j\rangle}{\eta\langle \tilde{S}_i\rangle} = 0$

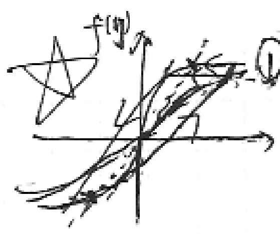
$E(\eta) = -J\sum_{\langle i,j\rangle}(\eta + \tilde{S}_i)(\eta + \tilde{S}_j) - h\sum_i(\eta + \tilde{S}_i)$

$= +JNd^2\eta^2 - h_{eff}\sum S_i$, $h_{eff} = h + 2Jd\eta$.

remark:
~~2D~~ → order of 2
→ order of 1

$Z = e^{h_{eff}/T} + e^{-h_{eff}/T}$; $\langle S_i\rangle = \tanh\left(\frac{h_{eff}}{T}\right)$

$\Rightarrow h_{eff} = h + 2Jd\tanh\left(\frac{h_{eff}}{T}\right)$



① as $h = 0$. we have: $\eta = \tanh\left(\frac{2Jd}{T}\eta\right)$.

$\begin{cases} \frac{2Jd}{T} > 1. & \eta \equiv 0. \\ \frac{2Jd}{T} < 1. & \eta = 0, \pm\eta^* \end{cases} \qquad T_c \triangleq 2Jd$

② with $h \neq 0$. $\eta = \tanh\left(\frac{2Jd}{T}\eta + \frac{h}{T}\right)$

as $T \sim T_c$, $\eta \ll 1$.

$\Rightarrow \eta \simeq \frac{T_c}{T}\eta + \frac{h}{T} \Rightarrow \eta = \frac{h}{T - T_c}$. $\chi = \frac{\partial \eta}{\partial h} = \frac{1}{T - T_c}$ Curie Weiss Law

[question: how to understand the loop structure more efficiently]

25. 1-D Ising model. $(N \gg 1)$

1). $h = 0$. spontaneous magnetization

$\mathcal{H} = -J(N-1) + 2J\cdot M$. M = # of "boundary".

ass① min $E \Leftrightarrow M = 0$ no dislocation $T = 0$. $S = \ln 2$

ass①.1 for $T > 0$., $F = E - TS$ should be min $\qquad S = \ln[2\cdot C_N^M]$

where $\frac{M}{N} \triangleq x \Rightarrow x = \frac{1}{e^{2J/T} + 1}$ $\qquad \simeq \ln 2 + x\ln x + (1-x)\ln(1-x)$

have a trivial limit: $x < \frac{1}{N}$. $\Rightarrow N > e^{2J/T} \Rightarrow T > \frac{2J}{\log N}$ $\frac{2J}{\ln N}$

[remark: when $x > 1/2$ this might be more interesting to talk about]

24. 2) general case $h \neq 0$.

formally $Z = \sum e^{-\mathcal{H}(S_i)} \Rightarrow \sum e^{\frac{hS_1}{2T} + \frac{hS_N}{2T}} (e^{\sum \cdots})$ → $\prod (e^{\cdots})$.

$= \mathrm{tr}\, M^N$. $\quad M = \begin{pmatrix} e^{(J+h)/T} & e^{-J/T} \\ e^{-J/T} & e^{(J-h)/T} \end{pmatrix}$

① transfer matrix M : $t \to t+\Delta t$.

$M(\Delta t) = \langle t_{i+1} | e^{-H/T} | t_i \rangle$ $\quad \boxed{T \sim \frac{\hbar}{\Delta t}. \text{ sounds profound}}$

$\det(M - \lambda) = 0$ gives us. $\lambda_\pm = e^{J/T} \left[\cosh\left(\frac{h}{T}\right) \pm \sqrt{\sinh^2\left(\frac{h}{T}\right) + e^{-4J/T}} \right]$

ass ① $N \gg 1$, $Z \approx \lambda_+^N$ $\quad$ (in fact $= \lambda_+^N + \lambda_-^N$).

$F = -T \ln Z = -TN \ln \lambda_+$

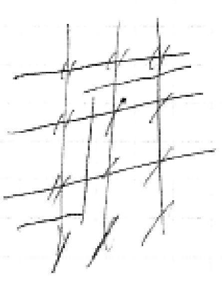
$\eta = \frac{1}{N} \sum S_i = \frac{T}{NZ} \partial_h Z = \frac{\sinh(h/T)}{\sqrt{\sinh^2\left(\frac{h}{T}\right) + e^{-4J/T}}} \overset{h\to 0}{\sim} \frac{h}{T} e^{2J/T}$

$\chi = \frac{\partial \eta}{\partial h} \underset{h\to 0}{=} \frac{1}{T} e^{2J/T}$.

$\Rightarrow$ for 1-D Ising model, $T=0$ serves as Curie point.

24.75. Ising model in 2-D.

$T_c = 2.27 J$ (Onsager)



$h = 0$, $\varepsilon_{++,--} = -J$. $\varepsilon_{+-} = +J$

① calculate the entropy of a "wall"

$E(L) = 2J \cdot L$, just 1 "wall". $\Omega = \frac{1}{2} 4 N^{1/2} (2d-1)^L$.

$F(L) = E(L) - S(L) \cdot T = (2J - T\ln 3) \cdot L + \text{const}(N)$.

case $\begin{cases} T < 2J/\ln 3 & \text{walls suppressed} \to \text{ordered phase} \to \text{spontaneous magnetization} \\ T > 2J/\ln 3 & \text{walls proliferate} \to \text{disordered phase} \end{cases}$

yet this is $T_c \simeq 1.82 J$.

T_c (meanfield) $= 4J$. (no entropy).

25. blocking transformation (clustering them)

$N \to \frac{N}{2^d}$. $\begin{cases} J \to J' \\ h \to h' \end{cases}$

for $\xi = \infty$, $(J, h) = (J', h')$ fixed point give us 2nd order transition

e.g. 1-D

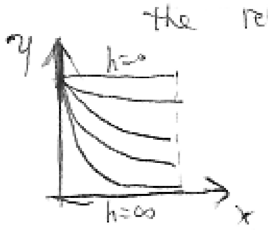
$Z = \mathrm{tr}\, M^N = \mathrm{tr}\,(M^2)^{N/2}$ $\quad M^2 = M(J', h')$. test J', h'.

$M^2 = \begin{pmatrix} e^{(2J+2h)/T} + e^{-2J/T} & e^{+h/T} + e^{-h/T} \\ e^{h/T} + e^{-h/T} & e^{-2J/T} + e^{2(J-h)/T} \end{pmatrix} = \text{Const} \cdot M\,?$

$\Rightarrow \frac{e^{(2J+2h)/T} + e^{-2J/T}}{e^{h/T} + e^{-h/T}} = \frac{e^{J+h/T}}{e^{-J/T}}$ and $\frac{e^{2(J-h)/T} + e^{-2J/T}}{e^{h/T} + e^{-h/T}} = \frac{e^{(J-h)/T}}{e^{-J/T}}$

parameter flow diagram.

the order is: fix h, get the relation between $e^{J/T}$ and $e^{-h/T}$.
the relation is given by the equation that we received before.



$$\begin{cases} x = e^{-2J/T} \\ y = e^{2h/T} \end{cases}$$ (for our particular system).

26. Monte Carlo simulations ~~for~~ of statistical system.

① e.g. Ising model : $Z = \sum e^{I[S_i]}$. $\{S_i\} \in \{ S_i \ 10\times10\times10\} \to 2^{10^3} \approx 10^{3\times100}$.
generate "relevant" spin configurations (Markov chain).

② Monte Carlo evolution "time" counting the updates

$$\langle O(S_i) \rangle = \frac{1}{|T_2 - T_1|} \sum_{T} O(S_i)$$

So we "randomly" pass through some states and average them, sort of average all possible states
that's how we handle "stationary" distribution.

Metropolis update : 1). $\{S_i\} \to E_i$

2). generate a candidate update $\{S_i\}$, E_i'

3) accept or reject. $p(i \to i+1) = \begin{cases} 1 & \Delta \le 0 \\ e^{-\Delta/T}, & \Delta > 0 \end{cases}$

$$\Rightarrow \frac{p(i\to i+1)}{p(i+1 \to i)} \equiv e^{-\Delta/T} \text{ all the time}$$

③ $\begin{cases} \text{hot start : random spin config} \\ \text{cold start : all spin up or down} \end{cases}$ $\Longrightarrow$ when they are converging: $E_{hot}(T) \approx E_{cold}(T)$.

"heatbath" algorithm, 2d model : update only $(x+y) \equiv 1 \pmod 2$ or $0 \pmod 2$ vertex
and no accept or reject, only follow from Gibbs distribution.

$\boxed{\text{Question: sign problem ?}}$

27. fluctuations. E. δE. (grand canonical potential).

① for $F = \frac{\sum f_i}{N}$. $\delta F = \frac{\delta f}{\sqrt{N}}$. for $(\delta F)^2 = \frac{\overline{(\sum f_i)^2} - (\overline{\sum f})^2}{N^2} = \frac{1}{N}(\overline{f^2} - (\bar f)^2) = \frac{(\delta f)^2}{N}$

Some algebra in Page 11. $\Rightarrow$

② $(\delta N)^2 = T \partial_\mu \langle N \rangle = T\partial_\mu \left(\frac{T}{Z_G} \partial_\mu Z_G\right)$.

☆ for ideal gas : $\langle N \rangle = e^{\mu/T} Z_{1p} = e^{\mu/T} \frac{V}{\lambda_T^3}$.

$$\Rightarrow (\delta N)^2 = \langle N \rangle$$

!! remark : we have 2 different definitions of μ, for fixed N, μ is fixed, which could be calculated from the expression of P13. yet for grand-canonical ensembles, μ is a variable for which we use ~~to~~ connect different N states

☆. Poisson distribution : $P_N = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle} = \frac{\lambda^N}{N!} e^{-\lambda}$ (should not use $\langle N \rangle$!!)

binomial distribution : $P_N = C_{N_{tot}}^{N} w^N (1-w)^{N_{tot}-N} \approx \frac{1}{N!} (N_{tot})^N w^N (1-w)^{N_{tot}}$

$w = \frac{N}{N_{tot}} \ll 1$

$$\longrightarrow \frac{(w \cdot N_{tot})^N}{N!} e^{-wN_{tot}}$$

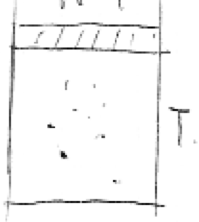
26

reproduce Gaussian distribution:

$$\lg P_N = N\lg\langle N\rangle - N\lg N + N - \langle N\rangle \qquad \Delta N \triangleq N - \langle N\rangle$$

$$\approx -\frac{(\Delta N)^2}{2\langle N\rangle} \quad \text{Gaussian ?}$$

$N_{tot} = \text{const}$



③ fluctuation of the volume V

Hamiltonian for the piston $\mathcal{H} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}ky^2$

pressure $\Leftrightarrow$ gravity equilibrium $\quad k = A\left(-\frac{\partial p}{\partial y}\right) = A^2\left(-\frac{\partial p}{\partial V}\right)_T \quad \frac{k}{m} \triangleq \omega^2$

$\langle E\rangle = 2\langle E_{kin}\rangle = M\omega^2(\delta y)^2 = \left(-\frac{\partial p}{\partial V}\right)(\delta V)^2$

and on the other side. $\langle E\rangle = \frac{1}{2}T + \frac{1}{2}T = T$.

$$\Rightarrow (\delta V)^2 = -T\cdot\left(\frac{\partial V}{\partial p}\right)_T \overset{\text{ideal}}{=\!=} \frac{V^2}{N}$$

④. fluctuation of pressure P. (tricky!)

$p = \frac{F}{A}$ where fluctuation could be treated as coming from $F \propto N$.

observation time τ is also a good dimension to think of,

In fact, $(\delta p)^2 \propto \frac{1}{\tau} \Rightarrow$ diverge as we vary the observation time quantity.

04/12/2017.

28. stochastic process

① random function of time $f(t)$. $\quad \Delta f = f - \langle f\rangle$ average over possibility

stationary process: $\left.\begin{aligned}\langle f(t)\rangle &= \langle f\rangle \\ \langle \Delta f(t)^2\rangle &= \langle(\Delta f)^2\rangle\end{aligned}\right\}$ independent on time

two variable, possibility and operator. (quan operators) not

Correlation: $\langle \Delta f(t)\,\Delta f(t')\rangle = K_f(t-t')$. (why ??)

$$\begin{cases} ① \text{ even pairing } K_f(t-t') = K(t'-t). \\ ② \ K_f(\tau\to 0) = (\delta f)^2. \\ ③ \ K_f(\tau\to\infty) = 0. \quad \text{for sufficient large } \tau \gtrsim \tau_{corr}\end{cases}$$

② Fourier - transformation

$$\begin{cases}\Delta f(t) = \int d\omega\, f(\omega)\, e^{-i\omega t} \\ f(\omega) = \int \frac{dt}{2\pi}\,\Delta f(t)\, e^{i\omega t}\end{cases} \qquad \left(\langle f(\omega)\rangle = \int\ldots\langle\Delta f(t)\rangle = 0\right)$$

question: where the integration form?

$$\Rightarrow \langle f(\omega)\cdot f^*(\omega')\rangle = \underset{\substack{\uparrow \\ \text{~~spontaneous~~ stationary stochastic process}}}{\delta(\omega-\omega')}\ \underbrace{\int e^{-i\omega\tau} K_f(\tau)\frac{d\tau}{2\pi}} \triangleq S_f(\omega)$$

$$\Rightarrow (\delta f)^2 = 2\int_0^\infty d\omega\, S_f(\omega) \approx 2\Delta\omega\, S_f(\omega)$$

range of ω: $|\omega| \leq \omega_{max} \sim \frac{1}{\tau_{corr}}$

① white noise:

$$S_f(\omega) = S_f(0) = \text{const}$$

$$K_{wn}(\tau) = 2\pi S_f(0)\,\delta(\tau)$$

or for $|\omega| \leq \omega_{max}$

$$K = 2 S_f(0)\,\frac{\sin(\omega_{max}\tau)}{\tau}$$

④. Langivan equation

(1) ① $m\ddot{q} + kq = F_{det}(t) + F_{stoch}(t)$ deterministic + random force

$\Rightarrow m\ddot{q} + \eta\dot{q} + kq = F_{det}(t) + \Delta F$

$\Rightarrow$ solution $q_{deterministic} + \Delta q$. $\Rightarrow q_\omega = \overline{\Delta q} = \dfrac{\overline{\Delta F}}{m(\omega_0^2-\omega^2) - i\omega\eta}$

$\Rightarrow S_q(\omega) = \dfrac{S_F}{m^2(\omega_0^2-\omega^2)^2 + \eta^2\omega^2}$

$\Rightarrow \langle \delta q\rangle^2 = 2\int_0^{\infty} \dfrac{S_F(\omega)\,d\omega}{m^2(\omega_0^2-\omega^2)^2+\eta^2\omega^2} \overset{\omega\approx\omega_0}{\approx} 2\int_0^{\infty}\dfrac{S_F(\omega)\,d\omega}{m^2(2\omega_0)^2(\omega-\omega_0)^2+\eta^2\omega^2}$

$= \dfrac{2S_F(\omega_0)}{\eta^2\omega_0^2}\cdot\dfrac{\eta}{2m}\int_{-\infty}^{+\infty}\dfrac{dx}{x^2+1} = \dfrac{\pi}{\eta k} S_F(\omega_0)$.

this is a very essential assumption

if this part is comparable to the main part of motion then:

$\langle E_{kin}\rangle = T/2 = \frac{k}{2}\langle(\delta q)^2\rangle \Rightarrow S_F(\omega_0) = \frac{\eta}{\pi}T \approx S_F(\omega)$ $|\omega-\omega_0| \ll \frac{1}{\tau_{corr}}$ or ω?

$\eta = \frac{\pi}{T}S_F(0) = \frac{\pi}{T}\int_{-\infty}^{+\infty}\frac{dt}{2\pi}\langle\Delta F(0)\,\Delta F(t)\rangle$

(2) ② η-large:

$\eta\dot{q} = F_{det} + \Delta F$.

$\Rightarrow \Delta q = \frac{1}{\eta}\int_0^t \Delta F(t')\,dt'$.

$\langle(\Delta q(t))^2\rangle = \frac{1}{\eta^2}\int_0^t\int_0^t dt_1 dt_2 \langle\Delta F(t_1)\Delta F(t_2)\rangle$

$= \frac{1}{\eta^2}\int_0^t\int_0^t \cancel{k_F}\, 2\pi S_F\,\delta(t_1-t_2) = 2\frac{T}{\eta}\cdot t \overset{\Delta}{=} 2Dt$

look at upper side and ②

$D = T/\eta$

⑤ Boltzmann distribution of Brownian particles

mg + collision + random motion

$n(h) = n_0\exp[-mgh/T]$

diffusion from high → low density

$\vec{j}_{dr} = n\vec{V} = -n\cdot\frac{mg}{\eta}\hat{z}$ (Newton)

$\vec{j}_{diff} = D(-\nabla n)$.

$\Rightarrow D = T/\eta$, $\partial_t n + \nabla(n\vec{v}) = 0$?

⑥. V. R. circuit

$q \to Q$.

$\dot{q} \to \dot{Q} = I$.

$F \to V$

$\eta \to R$. $(RI^2 = \dot{E} = \eta v^2)$.

$\Rightarrow \boxed{S_V = \frac{R}{\pi}T}$ Nyqist thermo noise.

$\langle(\delta V)^2\rangle_{\Delta\nu} = 4\cdot R\cdot T\cdot\Delta\nu$

$\langle(\delta I)^2\rangle_{\Delta\nu} = \frac{4}{R}T\cdot\Delta\nu$

⑦. generalized susceptibility

$S_F = \frac{\eta}{\pi}T$.

$\frac{dW}{dt} = V\cdot F = -\eta v^2$

$F(\omega) = \chi(\omega)\,q_\omega$

Reχ conserves energy, Imχ dissipate energy.

$W_\omega = \int_{-\infty}^{+\infty}(-\dot{q})F\,dt = \mathrm{Re}[\,i\omega q_\omega F^*] = (\mathrm{Im}\chi)\cdot|q_\omega|^2$

$S_F(\omega) = \dfrac{\chi''(\omega)}{\pi\omega}T$

28.

⑧ QM case:

$$\frac{k(\delta q)^2}{2} = \langle E_{\text{potential}} \rangle = \frac{1}{2}\langle E \rangle = \frac{\hbar\omega}{4}\coth\left(\frac{\hbar\omega}{2T}\right) \qquad (\text{H.O.})$$

$$\Rightarrow S_F(\omega) = \frac{\hbar\chi''}{2\pi}\coth\left(\frac{\hbar\omega}{2T}\right), \qquad S_q = \frac{S_F(\omega)}{|\chi|^2}$$

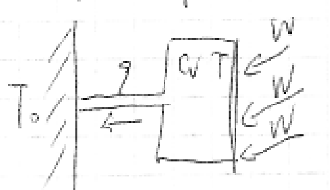
$$(\delta q)^2 = \frac{\hbar}{\pi}\int_0^\infty d\omega\, \frac{\chi''}{|\chi|^2}\coth\left(\frac{\hbar\omega}{2T}\right).$$

notice that $\frac{1}{\chi(\omega)} = \frac{\chi' - i\chi''}{|\chi|^2} \Rightarrow \frac{\chi''}{|\chi|^2} = \mathrm{Im}\left(-\frac{1}{\chi}\right)$

and $S_F = \begin{cases} \frac{\hbar\chi''}{2\pi} & \hbar\omega \gg T \quad \text{quantum fluctuation.} \\ \frac{\hbar\chi''}{\pi\omega} & \hbar\omega \ll T. \end{cases}$

$\boxed{\text{question: } S_F \text{ is dominated by sth. that is not relevant to itself? where is the concept been stolen?}}$

⑨ temperature fluctuation



$$T - T_0 = \frac{W}{q} \qquad \delta E = \sqrt{C_V}\cdot T \Rightarrow \delta T = \frac{\delta E}{C_V} = \frac{T}{\sqrt{C_V}}$$

~~single~~ signal vs noise ≥ 1. $\quad W \geq W_{\text{sens}} = q\cdot\delta T = \frac{qT_0}{\sqrt{C_V}}$

$$\frac{d}{dt}(C_V T) + q(T - T_0) = W_{\text{measure}} + \Delta W_{\text{stoch}} \quad (\text{Langevin equation})$$

$$\Rightarrow C_V(-i\omega)T_\omega + qT_\omega = W_\omega$$

$$\Rightarrow T_\omega = \frac{W_\omega}{-i\omega C_V + q}. \qquad (C_V)_{\max} = \frac{q}{\omega}.$$

(1) $\Rightarrow W_{\text{sens}} \geq \frac{qT_0}{\sqrt{q/\omega}} = T_0\sqrt{q}\sqrt{\omega}$

(2) $\Rightarrow (\delta T)^2 = 2\int d\omega\, S_T = \frac{\pi}{q\omega}S_W$, also $(\delta T)^2 = \frac{T^2}{C_V}$. (calculation twice)

$$\Rightarrow S_W = \frac{q}{\pi}T_0^2 \Rightarrow \sqrt{(\delta W)^2_{\Delta\nu}} = 2\sqrt{q}\sqrt{\Delta\nu}\cdot T_0.$$

$$\Rightarrow \frac{\delta W_{\text{sense}}(\Delta\nu)}{W_{\text{sense-full}}} \propto \sqrt{\frac{\Delta\nu}{\nu}} \qquad 1-2 \text{ order of magnitude.}$$

2) ~~Kinematics~~ Kinetics

① def. single-particle ~~prop~~ probability density $w(p, q, t)$.

$$N_{\text{total}} = \int w\, d^3p\, d^3q.$$

ass1. dependent distribution, dilute. or multi-collision. or non-interacting.

②. non-interacting particles:

continuity: $\partial_t w + \nabla_q\cdot\vec{j}_q + \nabla_p\cdot\vec{j}_p = 0. \quad \begin{cases} \vec{j}_q = w\cdot\dot{\vec{q}} \\ \vec{j}_p = w\cdot\dot{\vec{p}} \end{cases}$

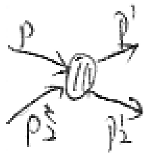
ass 2. by making use of Hamiltonian equation. we have:

$$\partial_t W + [W, H]_{classical} \equiv 0 \Rightarrow d_t W. \quad \boxed{\text{Liouville theorem}}$$

the probability density doesn't change along evolution.

③. the interacting picture:

ass 3. $$\frac{dW}{dt} = \frac{\text{collisions / unit}}{\text{time}} = \int d^3p' \, [\, T_{p \leftarrow p'} \cancel{(p')} W(p') - T_{p' \leftarrow p} W(p) \,]$$



$T_{p \to p'} = T_{p' \to p}$ when scattering is reversible.

Brownian motion of heavy particles mixture in light-molecular gas

also, for fermion, if $T_{p \to p'} = T_{p' \to p}$, then the formula are the same. also Boson,

~~$W(p) \Rightarrow W(p)$~~ i.e. $W(p) \to W(p)\,(1 \pm W(p))$.

④. Boltzmann equation ($2 \to 2$ collision, dilute gas regime)

ass 3' $$\frac{dW(\vec p)}{dt} = \int \frac{dp' \, dp_2}{dp_2'} \cdot \delta(p + p_2 - p' - p_2') \, T_{pp_2 \to p'p_2'} \cdot [W(p) W(p_2) - W(p') W(p_2')]$$

and again we ask for equilibrium:

$$\log W + \log W_2^{\xi} = \log W' + \log W_2'. \quad \text{as } \Sigma p - \Sigma p_f = 0.$$

(?) also take energy into account, we reach:

$$\log W(\vec p) = A\varepsilon + \vec b \cdot \vec p + C$$

let $\begin{cases} \langle \vec p \rangle = m\vec V \\ \langle \varepsilon \rangle = \frac{3}{2} T. \\ \int W(\vec p)\, dp = 1. \\ \varepsilon = \frac{p^2}{2m}. \end{cases}$ $\Rightarrow W(\vec p) = \exp\left[- \frac{(\vec p - m\vec V)^2}{2mT} \right]$

Boltzmann equation

⑤. entropy analysis: $S = - \Sigma W_m \ln W_m$.

formal. $$H = \int dp \, dq \; W(p) \ln W(p)$$

$$d_t H = \int dp^3 dq^3 \; \underset{\sim\sim\sim}{d_t W(p)} \, (\ln W(p) + 1). \quad \text{(use the previous expression)}$$

$$= \int dp^3 \, dq \, (\ln W(p) + 1) \int dp_2^3 \, d^3p_1' \, d^3p_2' \; \delta(p_0 + p_2 - p' - p_2') \dots$$

$$\xrightarrow{\text{Symmetry}} \frac{1}{4} \int \Pi \, d^3p \cdot \delta(\dots) \cdot T_{pp_2^{\xi} \to p'p_2'} \, (W_1 W_2 - W_1' W_2') [\ln(W' W_2') - \ln(W W_2)] \leq 0.$$

i.e. the entropy is increasing all the time with the help of collision

ass 4' violation: $H(t)$ is the local maximum refers to our assumption of "independent" distribution holds.

⑥. the equilibrium analysis based on ⑤:

0th-order-approximation: no dissipation $\Leftrightarrow$ no temperature fluctuation

$$\left.\begin{array}{l} W_1' W_2' = W_1 W_2 \\ p_1 + p_2 = p_1' + p_2' \\ \varepsilon_1 + \varepsilon_2 = \varepsilon_1' + \varepsilon_2' \\ 2 \text{ by } 2. \end{array}\right\} \text{ sth. not specific in equilibrium}$$

30.

trick here is: we try to ~~sp~~ seperate $w(p)$ around d^3q.

(1), potential is uniform $\longrightarrow F = \cancel{\text{const}}\,0 \longrightarrow \dot{\vec{p}} = 0$.

$\Rightarrow \partial_t W(q,p,t) + \dot{\vec{q}} \cdot \partial_{\vec{q}} w = (\text{collision integration}) = d_t W$.

then we look at:

$$\int d\vec{p}^3 \begin{pmatrix} 1 \\ m\vec{v} \\ \varepsilon \end{pmatrix} d_t W = \int dp_2 dp_1' \ldots (\quad) \cdots \quad (\text{expression in } ④)$$

$$\xrightarrow[\text{concerns.}]{\text{symmetry}} 0$$

which tells us sth. is conserved around the "local" region.

(2). some general definitions:

$$\begin{cases} n(q,t) = \int dp^3\, w. \qquad \rho = n \cdot m. \\ \vec{V} = \frac{1}{n} \int d^3p\, w \cdot \frac{\vec{p}}{m} \\ \xi = \frac{1}{n} \int d^3p\, w \frac{p^2}{2m}. \end{cases}$$

$$\begin{cases} \text{stress tensor: } \Pi_{ij} = \int d^3p\, w(p,q,t)\, \overset{mV_iV_j}{\cancel{\frac{p_ip_j}{m}}} \quad \text{momentum flux} \\ \text{energy flow: } \vec{Q} = \int d^3p\, w(p,q,t) \frac{1}{2} m v^2 \vec{v} \end{cases}$$

equations:

$$\begin{cases} \partial_t n + \partial_{\vec{q}} \cdot (n\vec{V}) = 0 \\ \partial_t (\rho \cdot \vec{V})_i + \partial_{q_j} \Pi_{ij} = 0. \\ \partial_t \xi + \partial_{\vec{q}} \cdot \vec{Q} = 0. \end{cases}$$

(~~dich~~ trick from formula: it seems that he used $\vec{V} = \vec{V}(q,t)$. rather than simple $\vec{v} = \frac{\vec{p}}{m}$) $(= \frac{d\vec{q}}{dt})$.

$\Rightarrow$ the distribution function thus have this form:

$$W(q,p,t) = W_0(\vec{p} - m\vec{V}(q,t)) = \exp\left[- \frac{(\vec{p} - m\vec{V}(q,t))^2}{2mT}\right] \quad \underline{\underline{\text{Local}}}$$

sort of fluctuation.

(3). $\Pi_{ij} \xrightarrow{\vec{v} = \vec{V} + \vec{u}} mn \cdot (V_iV_j + \langle u_iu_j \rangle)$.

$\xrightarrow[\text{analysis}]{\text{random}} \rho (V_iV_j + \frac{1}{3} \langle u^2 \rangle \delta_{ij})$.

$= \rho V_iV_j + \delta_{ij} \cdot \text{pressure}$. (use $\frac{1}{3}\rho \langle u^2 \rangle = T \cdot n = \text{pressure}$).

and $\xi = \frac{mV^2}{2} + \frac{1}{2} m \langle u^2 \rangle = \frac{1}{2} mV^2 + \bar{\varepsilon} \leftarrow$ true internal energy.

$\vec{Q} = \vec{V} (\bar{\varepsilon} + \frac{1}{2} mV^2 + \text{pressure})$.

⑦ small deviation from equilibrium.

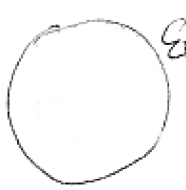
$W = W_0 + \delta W$.

$\Rightarrow \partial_t \delta W = -\vec{v} \cdot \partial_{\vec{q}} W_0 - \vec{F} \cdot \partial_{\vec{v}} W_0$.

and let $-\frac{\delta W}{\tau} \triangleq \delta(\frac{d}{dt} W_0) = \int \cancel{dp_1} dp_2 dp_1' dp_2' \; \delta(p_1 + p_2 - p_1' - p_2') \; w_1 w_2 (\frac{\delta w_1}{w_1} + \cdot 3 \cdot \text{terms})$.

$\Rightarrow$ called relaxation time approximation.

⑧ electron and Fermion statistics



ε_F

e – e scattering: ε_F.

e – i scattering: $\varepsilon_{final} \approx \varepsilon_F$

for $M_i \gg m_e$

⑨. stationary + uniform distribution:

ass

$$\frac{d}{dt} w \approx \frac{\partial}{\partial t}^{0} + 0 + \vec{F}\cdot\frac{\partial}{\partial \vec{p}} w_0 = -\frac{\delta w}{\tau}$$

uniform force field, $\vec{F} = $ const. also isotropic distribution.

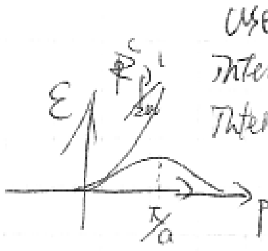
$$\Rightarrow \frac{\partial}{\partial \vec{p}} w_0 = \frac{\partial w_0}{\partial E}\cdot\vec{v} \longrightarrow \text{Boltzman / Fermi / Boson}$$

e.g. external E field. $\vec{F} = q\cdot\vec{E}$:

$$\begin{cases} \delta w = -\tau\cdot(\vec{F}\cdot\vec{v})\frac{\partial w_0}{\partial E} \\ \vec{j} = q^2\tau\int dp^3\ \vec{E}\cdot\vec{v}\vec{v}\left(-\frac{\partial w_0}{\partial \varepsilon}\right) \quad \left(= \oint q\langle\vec{v}\rangle\right) \end{cases}$$

$$= \frac{q^2\tau}{m^2}\vec{E}\ \frac{4\pi}{3}\int dp\cdot p^2\cdot p^2\left(-\frac{\partial w_0}{\partial \varepsilon}\right) = \sigma\vec{E}$$

use fermion distribution, $\Rightarrow \sigma = \frac{q^2\tau}{m}\frac{g}{(2\pi\hbar)^3}\frac{4\pi}{3}p_F^3 = \frac{q^2\tau}{m}\cdot n_e$



interpretation: $\vec{V}_{drift} = \vec{a}_c\cdot\tau = \vec{j}/n\cdot q$

Interesting phenomena:

$V_{group} = \frac{\partial \varepsilon}{\partial \vec{p}}$ moving back and forth.

⑩. stationary + non-uniform distribution.

ass I.

$W_0 = \frac{g}{h^3}\, n\left(\frac{\varepsilon-\mu}{T}\right)$, μ and T are both function of $\vec{q}$.

then: $-\frac{\delta w}{\tau} = \frac{d}{dt} w \approx 0 + \vec{v}\cdot\nabla\mu\left(-\frac{\partial w_0}{\partial \varepsilon}\right) - q\nabla\phi\cdot\vec{v}\frac{\partial w_0}{\partial \varepsilon}$

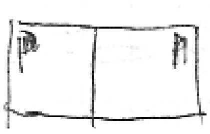
$$= -\frac{\partial w_0}{\partial \varepsilon}\vec{v}\cdot\nabla(\underbrace{\mu+q\phi}_{\text{electrochemical potential } \mu_e})$$

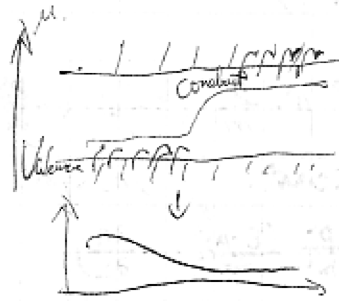
$$\longrightarrow \vec{j} = \sigma\vec{E}_{eff} \quad \begin{cases} \sigma = \frac{q^2\tau}{m}\cdot n_e \text{"="} \int dp^3\, q^2\tau\frac{\partial w}{\partial \varepsilon}\vec{v}\vec{v} \\ \vec{E}_{eff} = -\nabla\phi_{eff} = \vec{E} - \frac{1}{q}\nabla\mu \quad \left(= -\nabla\varphi - \frac{1}{q}\nabla\mu\right) \end{cases}$$

☆ two metal equilibrium:

$$\mu_e = \mu + q\phi = \mu_e'$$

e.g.





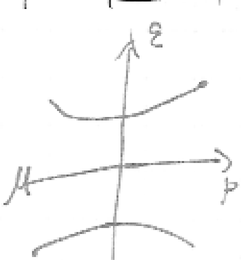
$n_e \propto \frac{1}{e^{\cdots}+1}$

$n_h \propto \frac{1}{e^{\cdots}+1}$

$\boxed{\text{tricky part: infinite sea anti-particle cancels out, finite quantity}}$

$$\begin{cases} p_v(x) = p_v(\infty)\, e^{-\frac{\phi_{eff}(\text{valence})}{T}} \\ n_c(x) = n_c(\infty)\, e^{-\frac{\phi_{eff}(\text{conduct})}{T}} \end{cases}$$

$\boxed{\text{Boltzmann !?}}$



yet, here $\phi_{eff}(\text{valance}) = \mu - \varepsilon_v + e\phi$

$\{\ \phi_{eff}(\text{conduct}) = \varepsilon_c - \mu - e\phi$

$$\Longrightarrow \vec{j} = \frac{q^2\tau}{m} n\, \vec{E}_{eff} \overset{\vec{E}_{eff}=0}{=\!=} \tau\frac{T}{m}\cdot q\nabla n = -q D\nabla n$$

$\boxed{\text{Einstein relation}}$ $\boxed{D = \mu T,\ \vec{F} = \frac{1}{\mu}\dot{\vec{x}}}$

$$= \Big(\underbrace{\frac{\tau}{m}\vec{F}}_{\text{drift}} - \underbrace{\frac{\tau}{m}T\nabla n}_{\text{diffusion}}\Big) q$$

$\Rightarrow$ continuum:

$$\partial_t n = \underbrace{\frac{T\tau}{m}}_{D}\Delta n - \nabla\cdot\left(n\frac{\tau}{m}q\vec{E}\right)$$

$\mu = \frac{\tau}{m} = \frac{1}{\eta}$ mobility

32.

ass/

(11) Local equilibrium

$d_t w = \partial_t w + (\vec{v}\cdot\nabla + \vec{F}\cdot\nabla_{\vec{p}}) w = \left(\begin{array}{l} \text{collision} \to \text{Boltzmann} \\ \text{relaxation} \end{array} \right).$

$w_0 = \frac{g}{h^3} \cdot n(\varepsilon) \quad (\text{or} = f(\frac{\varepsilon-\mu}{T}))$.

$\partial_\varepsilon w_0 = w_0 \cdot \frac{1}{T} f' \qquad \partial_T w_0 = -\frac{\varepsilon-\mu}{T^2} w_0 f'$

$\partial_\mu w_0 = -w_0 \frac{1}{T} f' \qquad$ "Local".

$\nabla w_0 = \frac{\partial w_0}{\partial \varepsilon} \cdot (-\nabla\mu - \frac{\varepsilon-\mu}{T}\nabla T)$.

$\Rightarrow \delta w = w - w_0 = \tau(\cdots)$

$= \tau \cdot \frac{\partial w_0}{\partial \varepsilon} \vec{v} \, (\underbrace{\nabla(\mu + q\phi)}_{} + \frac{\varepsilon-\mu}{T}\nabla T)$

electrochemical potential. $q\Phi$

☆. $\Rightarrow \vec{j}_q = \int d^3p \; q\vec{v}\,\delta w$.

$\stackrel{\Delta}{=} \sigma(-\nabla\Phi) + \sigma \cdot S(-\nabla T)$.

(1) then it's easy to see:

$\sigma = \tau q^2 \cdot \frac{g}{h^3} \cdot \frac{4\pi}{3} \int dp\, p^2 v^2 (-\frac{\partial n}{\partial \varepsilon})$.

trick/ass

make use of the former trick in Bosons / Fermions Statistics. P13

$\Rightarrow \sigma = \tau \cdot q^2 \cdot \boxed{\frac{g}{(2\pi\hbar)^3} \cdot \frac{4\pi}{3} p_F^3} \cdot \frac{1}{m} \quad ne$.

(2). thermo e.m.f. S.

$\sigma S = \int \cdots\cdots$

$= \frac{\pi^2 T}{2\varepsilon_F} \frac{q\tau}{m} \cdot \frac{g}{(2\pi\hbar)^3} \cdot \frac{4\pi}{3} p_F^3 = \frac{\pi^2 T}{2\varepsilon_F} \cdot \frac{\sigma}{q}$

(12). flux of heat.

$\vec{J}_h = \int dp^3\, \delta w\, \vec{v}\, (\varepsilon - \mu)$ ← substracting the static electric current

$0 = \partial_t (V C_V T)$

$+ \oint d\vec{A}\cdot\vec{J}_h$.

or $0 = C_V \partial_t T + \nabla\cdot\vec{J}_h$

$\stackrel{\Delta}{=} \sigma\Pi(-\nabla\Phi) + \mathcal{K}(-\nabla T)$

where:

$\sigma\Pi = \tau \cdot q \cdot \frac{g}{(2\pi\hbar)^3} \cdot \frac{4\pi}{3} \int_0^\infty p^2 \cdot \frac{p^2}{m^2} \cdot (\varepsilon-\mu) \cdot (-\frac{\partial n}{\partial \varepsilon}) = T\,\sigma \cdot S$

symmetry of coefficient. / Onsager's theorem

Same trick

$\mathcal{K} = \tau \cdot \frac{g}{(2\pi\hbar)^3} \cdot \frac{4\pi}{3} \int p^2 \frac{p^2}{m^2} \frac{(\varepsilon-\mu)^2}{T} (-\frac{\partial n}{\partial \varepsilon})$.

$= \frac{\pi^2 T}{3} \cdot (\sigma / q^2)$

(13) Brownian (1 particle).

(1) $\partial_t w = -\vec{\nabla}\cdot\vec{I}_w$

$\vec{v} = \mu\vec{F} = \frac{1}{\eta}\vec{F}$.

$\vec{I}_w = D(-\nabla w) + \frac{w}{\eta}\vec{F}$ (along $\vec{v}$)

$= \frac{1}{\eta}(-\nabla U)$

meta stable
↓
leak

and we have Einstein relation: $D = \frac{T}{\eta}$

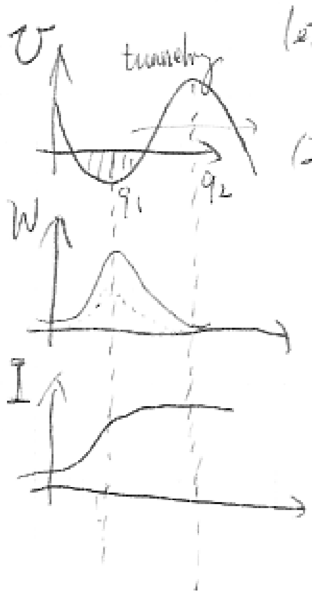
$\Rightarrow I_w = -\frac{1}{\eta}(T\nabla w + w\nabla u)$

and notice that $m\ddot{x} + \eta\dot{x} = 0 \Rightarrow \tau_{\max} \sim \frac{m}{\eta}$ momentum relaxation

$\Rightarrow \partial_t w = \frac{1}{\eta}(T\Delta w + \nabla(w\nabla u))$

let $\partial_t w = 0 \Rightarrow$ Boltzmann equation

$w \propto \exp(-u/T)$



(2). Krammers problem: leaking / tunneling / escaping / metastable state

$W = \int dq\, w(q)$ probability to stay in meta state

$d_t W = -\frac{1}{\tau} W$. decay rate

$I_w = -\frac{T}{\eta}\nabla w - \frac{1}{\eta} w\nabla u$

$= -\frac{T}{\eta} e^{-u/T}\nabla(w e^{+u/T})$

$\Rightarrow I_w \int_{q_1}^{q_2} e^{u(q)/T} dq \approx -\frac{T}{\eta} w e^{u/T}\Big|_{q_1}^{q_2}$ ☆

$\approx \frac{T}{\eta} w(q_1) e^{u_1/T}$

⇒ make quadratic approximation, we reach:

$I_w|_{\text{near } q_2} = \beta\, w(q_1)\left(\int_{q_1}^{q_2} dq\, e^{(u(q)-u_1)/T}\right)^{-1}$

$u(q) \approx u(q_2) - \frac{1}{2}K_2(q-q_2)^2$

$\approx u(q_1) + \frac{1}{2}K_1(q-q_1)^2$

~~$W|_{\text{near } q_1}$~~ $\approx \frac{T}{\eta} w(q_1)\left(e^{u(q_2)-u(q_1)}\int dq\, e^{-\frac{1}{2}K_2(q-q_2)^2}\right)^{-1}$

$= \frac{T}{\eta} w(q_1)\cdot e^{-\Delta u/T}\cdot\sqrt{\frac{K_2}{T}}\frac{1}{\sqrt{2\pi}}$

$W|_{\text{near } q_1} \approx \int dq\, w(q_1)\, e^{-\frac{1}{2T}K_1(q_1-q)^2} = w(q_1)\cdot\sqrt{\frac{2\pi T}{K_1}}$

$\Rightarrow I_w = w(q)\cdot e^{-\Delta u/T}\cdot\frac{\sqrt{K_1K_2}}{2\pi\eta}$ (then what's T?)

§4 Fokker-Planck equation

relaxation of momentum distribution heavy vs. light

heavy-heavy ⇇ vs heavy-light collision

$\partial_t w = \int d^3q\,[T_{p\leftarrow p+q}\, w(p+q) - T_{p-q\leftarrow q}\, w(p)]$

$\approx (q_\alpha \partial_{q_\alpha} + \frac{1}{2} q_\alpha q_\beta \partial_{q_\alpha}\partial_{q_\beta})\, T_{p-q\leftarrow q}\, w(p)$

$= \partial_{p_\alpha}(A_\alpha w + \partial_{p_\beta}(B_{\alpha\beta} w))$

$A_\alpha = \int d^3q\, q_\alpha T_{p-q\leftarrow q} = \frac{\sum q_\alpha}{\Delta t}$ collision sum $\approx \langle q_\alpha\rangle$

$B_{\alpha\beta} = \int d^3q\, q_\alpha q_\beta T_{p-q\leftarrow q} = \frac{\sum q_\alpha q_\beta}{2\Delta t}$ collision sum $= \langle q_\alpha q_\beta\rangle$

34.

?

$$\Rightarrow \partial_t w = \partial_{p_\alpha}[A_\alpha w + B_{\alpha\beta}\,\partial_{p_\beta} w].$$

$\hat{A}_\alpha + \partial_{p_\beta} B_{\alpha\beta}$, $B \cdot \delta_{\alpha\beta}$, stationary

stationary solution tells me : $w \propto \exp(-\frac{p^2}{2mT}) \Rightarrow A_\alpha = \frac{B}{MT} p_\alpha$

$$\Rightarrow \partial_t w = \partial_{p_\alpha}[\tfrac{B}{MT} p_\alpha w + B\partial_{p_\alpha} w].$$

Interesting problem :

$H_{MF} = -J[\sum \sigma_z^{(i)}\sigma_z^{(j)} + g\sum \sigma_x^{(i)}]$, $m_z = \langle \sigma_z \rangle$. matrix !! σ !!!

$= -J[\sum g m_z \sigma_z^{(i)} + g\sum \sigma_x^{(i)}] = -J\sum_i \vec{\Omega}\cdot\vec{\sigma}^{(i)}$ $\vec{\Omega} = (g, 0, g m_z)$

$Z_{MF}^{1\,spin} = tr(e^{-\beta H_{MF}}) = 2\cosh(\beta J|\Omega|)$

I feel that $\Sigma_E = \int d\Omega \, |\psi_{\vec{n}}\rangle\langle\psi_{\vec{n}}|$

$\sim tr(1)$. thus we could directly use tr. and also maintain H as matrix

i.e. $Z = tr(e^{-\beta H})$

$= tr(\int e^{-\beta H}|\psi\rangle\langle\psi| d\psi)$ exactly the form in QFT!

$= \int \langle\psi| e^{-\beta H}|\psi\rangle d\psi$