Homework 2

Phys 556: Solid State II
Assigned: Mar. 2, Due: Mar. 12

1 Time-ordered exponential (Problem 2.3 in Ref. [1])

Define the time-ordered exponential by:

$$T\left[e^{-\int_{t_a}^{t_b} dtA(t)}\right] = \lim_{N\to\infty} e^{-\epsilon A(t_N)}e^{-\epsilon A(t_{N-1})}\cdots e^{-\epsilon A(t_1)}e^{-\epsilon A(t_0)}, \tag{1}$$

where $\epsilon = \frac{t_b - t_a}{N}$ and $t_n = t_a + n\epsilon$. The time-ordered exponential can be expanded as a Taylor series:

$$T\left[e^{-\int_{t_a}^{t_b} A(t)}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{t_a}^{t_b} dt_1 \dots dt_n T\left[A(t_1)\dots A(t_n)\right] \tag{2}$$

By replacing the integrals in Eq. (2) with discrete sums, prove that Eqs. (1) and (2) are equal to order ϵ^3. (Assume the operator A is bosonic so that we do not need to worry about the minus signs in the definition of the time-ordering operator.)

2 Properties of the time-evolution operator

The time-evolution operator,

$$S(t_2, t_1) \equiv T\left[e^{-i\int_{t_1}^{t_2} V(t)dt}\right], \tag{3}$$

satisfies $|\psi_I(t)\rangle = S(t,t')|\psi_I(t')\rangle$, where the subscript I denotes the interaction representation with respect to the Hamiltonian $H = H_0 + V$. Prove the following properties:

1. $S(t,t) = 1$
2. $S^\dagger(t,t') = S(t',t)$
3. $S(t,t')S(t',t'') = S(t,t'')$

(These properties are nearly trivial to prove. But they are useful to remember.)

3 Gell-Mann-Low theorem

Let $H = H_0 + V$, where H_0 is simple enough that its eigenstates and eigenvalues are known. Define ϕ_0 to be the ground state of H_0 and let $\psi_I(t)$ denote the ground state of the full Hamiltonian in the interaction representation. Furthermore assume that in the distant past and far future, the interaction term is 0, i.e., $V(\pm\infty) = 0$. We can then define

$$\psi_I(-\infty) = \phi_0 \tag{4}$$

Then using the S matrix, $\psi_I(\infty) = S(\infty, -\infty)\phi_0$. We deduce that:

$$\langle \phi_0 | \psi_I(\infty) \rangle = \langle \phi_0 | S(\infty, -\infty) | \phi_0 \rangle \equiv e^{iL}, \tag{5}$$

which defines a phase e^{iL} that describes how the ground state evolves as the interactions are slowly turned up and then down.

The many-body Green's function is defined as:

$$G(\mathbf{k}, t - t') \equiv -i \langle \phi | T \left[\psi_{\mathbf{k}}(t) \psi^{\dagger}_{\mathbf{k}}(t') \right] | \phi \rangle, \tag{6}$$

which is implicitly defined in the Heisenberg representation, that is, the many-body ground state $|\phi\rangle$ is time-independent and the fields $\psi_{\mathbf{k}}(t)$ are regarded as operators that evolve according to the full Hamiltonian. We would like to rewrite the Green's function in terms of the states and operators in the interacting representation.

(a) Prove that the S matrix also transforms between the Heisenberg and interaction representations, that is:

$$O_H(t) = S(0, t) O_I(t) S(t, 0) \tag{7}$$

Using Eq. (4), it follows that $\phi = S(0, -\infty)\phi_0$.

(b) Using Eqs. (4), (5), and (7), derive the Green's function in terms of the interacting representation:

$$G(\mathbf{k}, t - t') = -i \frac{\langle \phi_0 | T \left[\psi_{I,\mathbf{k}}(t) \psi^{\dagger}_{I,\mathbf{k}}(t') S(\infty, -\infty) \right] | \phi_0 \rangle}{\langle \phi_0 | S(\infty, -\infty) | \phi_0 \rangle} \tag{8}$$

The placement of $S(\infty, -\infty)$ in the numerator does not matter because of the time-ordering operator. Eq. (8) is an example of the Gell-Mann-Low theorem.

4 Wick's theorem

Let us ignore the constant phase factor in the denominator of Eq. (8). Then we can use Eq. (2) and the definition of S in Eq. (3) to evaluate the Green's function perturbatively, that is,

$$G(\mathbf{k}, t - t', s, s') = \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} \int_{-\infty}^{\infty} dt_1 t_2 \dots dt_n \langle \phi_0 | T \left[\psi_{I,\mathbf{k},s}(t) V_I(t_1) V_I(t_2) \cdots V_I(t_n) \psi^{\dagger}_{I,\mathbf{k},s'}(t') \right] | \phi_0 \rangle, \tag{9}$$

where we have added the spin indices s and s'. The zeroth order ($n = 0$) term is the free-fermion Green's function that we derived in class:

$$G^0(\mathbf{k}, t, s, s') = -i \left[(1 - n_{\mathbf{k}}) \Theta(t) - n_{\mathbf{k}} \Theta(-t) \right] e^{-i\epsilon_{\mathbf{k}} t} \delta_{ss'}, \tag{10}$$

where $\epsilon_{\mathbf{k}}$ are the eigenvalues of H_0. Let $V_I(t)$ be the electron-electron interaction term:

$$V = \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \sum_{s,s'} \frac{4\pi e^2}{q^2} \psi^{\dagger}_{\mathbf{k}+\mathbf{q},s} \psi^{\dagger}_{\mathbf{k}'-\mathbf{q},s'} \psi_{\mathbf{k}',s'} \psi_{\mathbf{k},s} \tag{11}$$

Write the first order ($n = 1$) contribution to $G(\mathbf{k}, t, s, s')$ in terms of G^0 using Wick's theorem.

References

[1] "Quantum Many-Particle Systems" by John W. Negele and Henri Orland