Homework 5

Phys 556: Solid State II
Assigned: April 17, Due: April 30

1 Generalized Landau theory: $u < 0$ (Adapted from Exercise 11.3 in Ref. [1])

Consider the more general class of Landau theory where the interaction u can be negative:

$$f[\psi] = \frac{r}{2}\psi^2 + u\psi^4 + u_6\psi^6 - h\psi \tag{1}$$

(a) Show that for $h = 0, u < 0, r > 0$, the free energy contains three local minima, one at $\psi = 0$ and two others at $\psi = \pm\psi_0$, where

$$\psi_0^2 = -\frac{u}{3u_6} \pm \sqrt{\left(\frac{u}{3u_6}\right)^2 - \frac{r}{6u_6}}. \tag{2}$$

(b) Show that for

$$r < r_c \equiv \frac{u^2}{2u_6}, \tag{3}$$

the solution at $\psi = 0$ becomes metastable. Solve for r_c by setting $f[\psi_0] = f[0] = 0$. Show that, unlike the $u_6 = 0$ case discussed in class, ψ jumps discontinuously across the phase transition.

(c) Assume $r = a(T - T_c)$. Sketch the (T, u) phase diagram for $h = 0$ and fixed $u_6 > 0$.

(d) Consider $r = 0, h \neq 0$. Solve for ψ in the limiting cases $|h| \gg |u|$ and $|h| \ll |u|$.

(e) Argue that there are two critical lines in the $r = 0$ plane which cross at $h = u = 0$.

2 Coherent states (Adapted from Ref. [2])

Let $b^\dagger$ be a boson creation operator. Given a complex number, α, define the coherent state $|\alpha\rangle$ by:

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha b^\dagger}|0\rangle = e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}}|N\rangle, \tag{4}$$

where

$$|N\rangle \equiv \frac{1}{\sqrt{N!}}(b^\dagger)^N|0\rangle. \tag{5}$$

(a) Prove that $|\alpha\rangle$ is normalized.

(b) Prove that $|\alpha\rangle$ is an eigenstate of b with eigenvalue α.

(c) Prove that the set $\{|\alpha\rangle\}$ spans the space $\{|N\rangle\}$. (In other words, show that each $|N\rangle$, $N \geq 0$, can be written as a superposition of $\{|\alpha\rangle\}$.)

(d) Compute $\overline{N} = \langle\alpha|\hat{n}|\alpha\rangle$ and $\overline{N^2} = \langle\alpha|\hat{n}^2|\alpha\rangle$ and show that $\Delta N = \sqrt{\overline{N^2} - (\overline{N})^2} \ll \overline{N}$ when $\overline{N} \gg 1$. Therefore, in a coherent states with $\overline{N} \gg 1$, b and $b^\dagger$ behave approximately like classical variables, that is, $\langle[b, b^\dagger]\rangle \ll \overline{N}$.

3 Pair correlation function and superconducting instability

(a) The pair correlation function, which arises from the Cooper pair propagator, is given by:

$$\chi^{\mathrm{SC}}_{\omega_n,\mathbf{q}} = -\frac{T}{V}\sum_{\omega_m,\mathbf{p}} G_0(\mathbf{p}, i\omega_m)G_0(-\mathbf{p}+\mathbf{q}, i\omega_{-m+n}), \tag{6}$$

where $\omega_n = \frac{(2n+1)\pi}{\beta}$ is a Matsubara frequency (recall the Matsubara frequencies result from Fourier transforming the imaginary time correlation functions), $G_0(\mathbf{p}, i\omega_n) = (i\omega_n - \epsilon_{\mathbf{p}})^{-1}$ is the free particle propagator, $\beta = 1/T$ and V is the volume of the system. Using the sum formula:

$$\frac{1}{\beta}\sum_{\omega_n}\frac{1}{i\omega_n - \epsilon} = n_F(\epsilon), \tag{7}$$

where $n_F(\epsilon)$ is the Fermi distribution function (recall we proved Eq. (7) in class in the first half of the semester), prove that:

$$\chi^{\mathrm{SC}}_{\omega_n,\mathbf{q}} = \frac{1}{V}\sum_{\mathbf{p}}\frac{1 - n_F(\epsilon_{\mathbf{p}}) - n_F(\epsilon_{-\mathbf{p}+\mathbf{q}})}{i\omega_n - \epsilon_{\mathbf{p}} - \epsilon_{-\mathbf{p}+\mathbf{q}}} \tag{8}$$

(b) The static pair correlation function is given by the $i\omega_n \to 0$ limit:

$$\chi^{\mathrm{SC}}(\mathbf{q}) = \frac{1}{V}\sum_{\mathbf{p}}\frac{1 - n_F(\epsilon_{\mathbf{p}}) - n_F(\epsilon_{-\mathbf{p}+\mathbf{q}})}{-\epsilon_{\mathbf{p}} - \epsilon_{-\mathbf{p}+\mathbf{q}}} \tag{9}$$

Prove that when $\mathbf{q} = \mathbf{0}$, $\chi^{\mathrm{SC}}(\mathbf{q} = \mathbf{0})$ diverges as $T \to 0$ like $\ln\left(\frac{\omega_D}{T}\right)$, assuming that the pair interaction is only non-zero for a thin shell around the Fermi surface of width ω_D, where ω_D is the Debye frequency. (Hint: turn the sum into an integral over energy and assume that the density of states is constant over the region of integration.) (Note: this procedure yields $\chi^{\mathrm{SC}}(\mathbf{q} = \mathbf{0}) \approx \nu\ln\left(\frac{\omega_D}{T}\right)$. The two-particle propagator can be summed diagrammatically to yield a vertex term: $\Gamma = g\left(1 - g\chi^{\mathrm{SC}}(\mathbf{q})\right)^{-1}$. Thus, the $\mathbf{q} = \mathbf{0}$ vertex diverges when $T = T_c \equiv \omega_D e^{-1/(g\nu)}$, as we also derived in class. The diagrammatic derivation can be found in Sec. 6.4 of Ref. [3].)

(c) Now suppose the spin-up and spin-down Fermi surfaces are split by a Zeeman term, which can be implemented by a shift in the Fermi distribution function. Argue that $\chi^{\mathrm{SC}}(\mathbf{q})$ does not have a $T \to 0$ divergence. For which values of $\mathbf{q}$ will $\chi^{\mathrm{SC}}(\mathbf{q})$ be peaked? (Note: superconductors with finite momentum pairing are called Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) states.)

References

[1] "Introduction to Many-Body Physics" by Piers Coleman.

[2] "Modern Condensed Matter Physics" by Steven Girvin and Kun Yang.

[3] "Condensed Matter Field Theory" by Alexander Altland and Ben Simons.