Math Club Discussions

September 19, 2019

1 8/28/19: On Plane Curves

Using Lecture 12 of *The Mathematical Omnibus*.

1. Draw an exciting closed curve that is smooth. (Draw our own on the board.)
2. Smooth: No sharp corners.
3. Wiggling argument
   (a) Define a double point: Points of self-intersection.
   (b) What is a triple point? We don't want it on our curve, so wiggle it.
   (c) Define a double tangent: tangent at two **distinct** points.
   (d) what is a self-tangency: The two points coincide. We don't want it, so wiggle it.
   (e) What is a triple tangent? We don't want it on our curve, so wiggle it.
4. Question: What kinds of double tangents are there?
   (a) outer
   (b) inner
5. Count the number of your outer and inner double tangents and your double points. Call the number of inner double tangents T_- and the number of outer double tangents T_+ and the number of double points D.
6. Inflection points
   (a) Inflection Point: moving along the curve, one is turning left or right. the inflection points are where the direction of rotation changes.
   (b) Can you think of a smooth closed curve with exactly one inflection point.
   (c) What can you say in general? Ans: Number of inflection points is even.

7. Fabricius-Bjerre formula: $T_+ - T_- - \frac{1}{2}I = D$
   (a) Check $T_+ - T_- - \frac{1}{2}I$ if it is less than or greater than D.
   (b) (check with the curve drawn on the board)
8. Why is this true?
   (a) Choose a starting point and a walking direction. Shoot a laser in front of you and count the number of intersections of this laser with the curve in front of you. Let N be this number.
   (b) Start walking. The rest on our 3 problems (week 13) file.

2 9/18/19: Rotations and Braids

Using Arnold's book, a paper titled *Touching the $\mathbb{Z}_2$ in Three-Dimensional Rotations* by Vesna Stojanoska and Orlin Stoytchev, and online handout at Cambridge.

1. Rotations in $\mathbb{R}^3$ around an axis:
   (a) Q: Which parameters need to be specified to uniquely determine such rotations?
   A: Take a unit vector and an angle.
   (b) Q: Is this unique?
   A: No.
      i. $(\alpha, \vec{u}) = (-\alpha, -\vec{u})$
      ii. $(\pi, \vec{u}) = (\pi, -\vec{u})$
      iii. $(0, \vec{u}) = (0, \vec{v})$
   (c) Q: What is the space of all such rotations then?
   Write the correspondence $(\alpha, \vec{u}) \to \alpha\vec{u}$. What is this?
2. Rotations in $\mathbb{R}^3$ around a fixed point
   (a) These rotations are like moving a knob attached to a sphere.
   (b) Q: How can each rotation of this sphere correspond to some path on the other space?
   A: Think of such a rotation as a movie...
   (c) Complete rotations correspond to loops in the space of rotations.
3. The fundamental group of $SO(3)$.
   (a) Take loops starting and ending at a fixed point.
   (b) Loop 1 is deformable to loop 2 if you can physically move one to get the other.
   (c) How many different kinds of closed loops in $SO(3)$ that start and end at the origin?

(d) Therefore, we claim that it is isomorphic to $\mathbb{Z}_2$.

4. One other way to visualize it: Feynman's plate trick.

5. One possible proof: Use braids which was Dirac's idea.

   (a) Begin with a sphere with three strands attached to it.

   (b) Certain quotient group of the pure braid group correspond to loops in $SO(3)$. Claim that the fundamental group of $SO(3)$ is isomorphic to this quotient which happens to be $\mathbb{Z}_2$.

   (c) Demonstration using physical model.