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A marginal analysis framework to incorporate the externality effect of ordering perishables

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ABSTRACT

The exact analysis of optimal ordering policies for periodic-review perishable inventory systems is challenging because of their high-dimensional state space arising from multiple interrelated ordering decisions over many periods and age distributions of on-hand products. We develop a new marginal analysis framework to approximate this complex multiple-decision model into a single-decision model with an externality term, which internalizes the long-term impact of ordering decisions. Our externality-based approximation utilizes a constant base-stock policy; it is fast and easy to apply. Numerical experiments show that our approach provides state-dependent ordering amounts almost identical to the optimal dynamic programming-based policy.

1. Introduction

A large portion of perishable products (e.g., food, drugs, chemicals, blood products, and even fashion goods) are wasted every year. However, the analysis of inventory policies that better manage perishable products remains a challenging problem. The main challenge in analyzing perishable inventory models is their high-dimensional state space due to interrelated periodic ordering decisions over infinitely many periods and different age distributions of on-hand products. For simpler models, we may be able to use dynamic programming (DP). However, the exact analysis of perishable inventory models using stochastic DP is computationally expensive, rendering it intractable for models with large state spaces.

As noted in Karaesmen et al. [1], "The policy structures outlined in Fries [2] and Nahmias [3] [the first papers that developed models for perishable inventory] are quite complex; perishability destroys the simple base-stock structure of optimal policies for discrete review models without fixed order costs in the absence of perishability". Furthermore, the DP approach does not provide any insight into the form of the inventory-dependent (which we refer to as *state-dependent*) optimal policy.

Many researchers have thus sought effective heuristic methods [for comprehensive reviews, see, e.g., 1,4]. Among these heuristics, the constant base-stock (CBS) policy – despite its simplicity – is an excellent alternative to the optimal state-dependent policies [5–7]. Many state-dependent policies have also been proposed, among which two approaches have received increased attention, namely $L^{\natural}$*-convexity* [e.g., 8–10] and the *marginal cost accounting scheme* [e.g., 11–14]. The marginal cost accounting scheme utilizes marginal analysis, which provides an efficient algorithm for perishable inventory models. The key of the marginal cost accounting scheme is to develop an effective cost-balancing technique for the specific model under consideration, which is often not straightforward to identify.

We develop a marginal analysis framework that incorporates the *externality* effect – the indirect long-term impact of ordering decisions – on the average cost of a perishable inventory system. To our knowledge, the inclusion of the externality effect in a marginal analysis framework has not been employed in the inventory management literature, though it has been widely implemented to study economic concepts, including congestion pricing with application in airport runway capacity [15], airline route selection [16], and on-street parking [17]. Using this framework, we derive an approximate optimality condition for the general state-dependent policy. The approximate optimality condition is a recursive equation, which is still challenging to solve. However, by utilizing the properties of the CBS policy, we can reduce the externality

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effect into a fixed cost or benefit, representing the marginal external cost. Namely, under this approximation scheme, we can convert the original complex exact optimality condition into a simple approximate optimality condition in which only a single order amount for a given initial inventory level is involved. This single-decision condition is *almost* identical to the optimality condition for the newsvendor model, and hence, is easy to solve, for example, in a spreadsheet. This approach provides near-optimal solutions both for the average cost and the individual order amounts. In addition, our approach provides insight into the state-dependent characteristics of near-optimal ordering policies.

There is an abundance of near-optimal heuristics for perishable inventory systems in the literature. In this paper, our primary contribution is not to add one more element to this list but rather to provide a general framework to convert numerically intractable multi-decision stochastic dynamic inventory models to tractable single-decision models. Our framework is motivated by the density functional theory [18] and its local density approximation [19]—the most popular and successful computational physics and chemistry methods to convert multi-body problems into single-body problems. Another closely related method is Newton's iterative method to find a minimizer $\mathbf{x}^*$ of a function $f(\mathbf{x})$ starting from an initial guess $\mathbf{x_0}$. Our approach starts from $\mathbf{x_0}$ (a CBS policy) and moves toward $\mathbf{x}^*$ (an optimal/near-optimal state-dependent ordering policy), following the direction to minimize the total cost function. This simple approach may not always work, but when it does, it efficiently finds an optimal/near-optimal solution in a high-dimensional space.

We validate that our approximation based on the marginal analysis framework yields near-optimal state-dependent ordering policy for a perishable inventory system, in particular, the single-product periodic-review fixed lifetime inventory model with lost sales, no setup cost, and zero lead time (the *Nahmias model*), for which the DP solution is available [2,3]. We observe that the average costs we obtain using our approach are within 0.4% of those obtained by DP for all cases we investigated, and that the individual order amounts for different initial inventory levels also closely match those derived by DP. While finding the approximate optimal policy using marginal analysis framework takes up to a few minutes for all experiments (and can be implemented even in a spreadsheet for problems with short lifetimes), finding the optimal solution using DP algorithm takes significantly longer computation time.

In Section 2, we present our marginal analysis framework to derive the optimality condition of the ordering policy for an infinite-horizon inventory system with a single perishable product under a *general* cost function. In Section 3, we apply the framework to the Nahmias model to derive the approximate optimality condition by deriving the externality term for that model. In Section 4, we outline the algorithm to implement the marginal analysis framework and evaluate its accuracy (both with respect to the total inventory cost and the individual order amounts) through numerical experiments. We discuss insights generated by our method and managerial implications in Section 5. Finally, Section 6 summarizes our study. All proofs are provided in the appendix.

2. Marginal analysis framework

This section describes our marginal analysis framework to find the properties of an infinite-horizon inventory system that involves a high-dimensional state space: perishable products with different ages. Our objective is to derive the optimality condition of the state-dependent ordering policy for such inventory systems. The analysis we present in this section is general; the marginal analysis framework is applicable to any models that satisfy Eqs. (2) and (5) in Section 2.1. However, for the sake of explaining the physical meaning of equations in detail, we discuss a single-product periodic-review perishable inventory system operating under the first-in first-out (FIFO) issuing policy with zero lead time and lost sales. We do not consider any capacity constraints for the order or inventory. The dynamics in each period is as follows: a new order is placed, demand is fulfilled, perished products are discarded, and costs are incurred. Specifically, let $m\,(\geq 1)$ be the product lifetime, x_i be the number of units with $i \in [1, m]$ remaining lifetime periods, and D_j be the independent and identically distributed (i.i.d.) demand for period j. Then, Fig. 1 depicts the schematic illustration of the dynamics of this inventory system, where $x_1, \ldots, x_{m-1}$ represents the *initial inventory* (with different remaining lifetimes) and x_m represents the *order amount* (as lead time is zero, the order amount with lifetime of m periods is received immediately). See Table 1 for notations used in this paper.

Table 1
List of notations.

m	Product lifetime
x_i	Number of units with $i \in [1, m]$ remaining lifetime periods
x_m	Order amount
w_i	Inventory level with remaining lifetime of at most i periods
x	Total initial nventory; $x = w_{m-1}$
$\mathbf{x}^i$	Inventory vector with the remaining lifetime of at most i periods
$\mathbf{x}$	Initial inventory vector; $\mathbf{x} = \mathbf{x}^{m-1}$
$q(\mathbf{x})$	Order-up-to level for initial inventory vector $\mathbf{x}$
D_j	Independent and identically distributed (i.i.d.) demand for period j
D	Demand when period j is not specified
$L(q, \mathbf{x})$	One-period cost for the single ordering decision q and initial inventory vector $\mathbf{x}$
$L(q)$	Average total cost of the stationary model
q^*	Optimal inventory policy
q_h^*	Approximate optimal policy
q_c^*	Optimal CBS policy
c	Per-unit purchase cost
h	Per-unit holding cost per period
r	Per-unit shortage cost
θ	Per-unit wastage cost
$n_h(q)$	Expected number of units held per period
$n_s(q)$	Expected number of units in shortage per period
$n_w(q)$	Expected number of units wasted per period
$w_{ex}(q)$	Externality component per period
$A_i(\mathbf{x}^{i-1})$	Random variable for i-period effective demand
$\mathbf{X}^{q_c}$	Initial inventory random vector under the CBS policy q_c

For ease of representation, let $w_i = \sum_{j=1}^{i} x_j$ be the total inventory amount with the remaining lifetime of at most i periods and $x = w_{m-1} = \sum_{j=1}^{m-1} x_j$ be the total initial inventory. Similarly, let $\mathbf{x}^i = (x_1, \ldots, x_i)$ be the inventory vector with the remaining lifetime of at most i periods and $\mathbf{x} = \mathbf{x}^{m-1} = (x_1, \ldots, x_{m-1})$ be the initial inventory vector. Let $\Omega = \mathbb{R}_{\geq 0}^{m-1}$, a set of non-negative real numbers in an $(m-1)$-dimensional vector space; then $\mathbf{x} \in \Omega, \forall \mathbf{x}$.

Explaining the above perishable inventory model in a multi-echelon structure, the ith echelon contains the number x_i of products that will perish i periods in the future. At the end of each period, the remaining inventory in echelon 1 (i.e., x_1) is discarded. As we transition to the next period, the inventory in echelons $2, \ldots, m$ (i.e., $x_2, \ldots, x_m$) are moved to their corresponding next lower echelons $1, \ldots, m-1$ (i.e., $x_1, \ldots, x_{m-1}$). New inventory (equivalently, order amount x_m) enters the highest echelon m. Based on the FIFO issuing policy, demand is fulfilled using the inventory at echelon 1 (i.e., x_1) first and then at higher echelons. This model also assumes that all stock arrives new.

We characterize an ordering policy by its order-up-to level $q(\mathbf{x})$, a scalar-valued function of the initial inventory vector $\mathbf{x}$. With a slight abuse of notation, we denote q to represent either the order-up-to level $q(\mathbf{x})$ for a particular $\mathbf{x}$ or the policy $q(\cdot)$, a function of $\mathbf{x}$. When

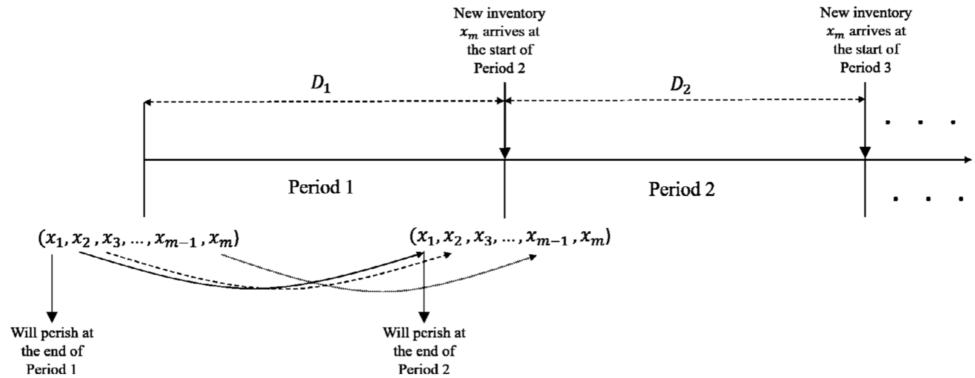


Fig. 1. The dynamics of the inventory system.

implementing the policy q, the order amount at the beginning of each period becomes $x_m = \max\{q(\mathbf{x}) - x, 0\}$, or simply $x_m = [q - x]^+$.

In summary, the state of the inventory system is $\mathbf{x} = \mathbf{x}^{m-1} = (x_1, x_2, \dots, x_{m-1})$, and the decision is the order amount x_m (which is determined by the order-up-level $q(\mathbf{x})$). In each period, the items with the shorter remaining lifetime are consumed first by demand, and the unused items with one period of remaining lifetime perish at the end of the period. Then, the system state in the next period is updated by reducing the lifetime of the remaining items by one.

We propose a stationary model of this problem based on the ensemble-average cost (taken over the initial inventory distribution), instead of its stochastic dynamic program (DP) model, which is known to be computationally difficult to solve due to the curse of dimensionality and dependence of decisions among different periods. Our stationary model incorporates the complexity of tracking inventory levels in infinite periods into the initial inventory distribution. The infinite-horizon DP model and the stationary model represent the same average total cost; DP calculates the time-average cost, and the stationary model calculates the ensemble-average cost.

When demand is independent and identically distributed (i.i.d.), we can define each period's initial inventory $\mathbf{X}$ as a non-negative *random vector* following a stationary distribution $f^q_{\mathbf{X}}(\cdot)$ given the policy q.[1] Let $L(q, \mathbf{x})$ be the one-period cost associated with the single ordering decision q when the initial inventory $\mathbf{x}$ is observed at the beginning of the period. Then, the average total cost of the stationary model follows:

$$L(q) = \mathbb{E}_{\mathbf{X}}[L(q, \mathbf{X})] = \int_{\Omega} L(q(\mathbf{k}), \mathbf{k}) f^q_{\mathbf{X}}(\mathbf{k}) d\mathbf{k}, \tag{1}$$

which is a functional of the policy $q = q(\mathbf{x}), \forall \mathbf{x} \in \Omega$.

2.1. Optimality condition

Let $q^* = \arg\min L(q)$ be the minimizer of $L(q)$ (i.e., q^* is the optimal inventory policy). When $L(q)$ is convex (which is the case for many inventory models, including the perishable inventory model discussed in Section 3), an inventory policy with the order amount $x_m = [q^* - x]^+$ should minimize $L(q)$ under the constraint $q(\mathbf{x}) \geq x$. Hence, it suffices for our purpose to find the (unconstrained) minimizer q^* of $L(q)$, which satisfies the optimal functional derivative condition: $\delta L(q)/\delta q(\mathbf{x}) = 0, \forall \mathbf{x} \in \Omega$. Therefore, q^* satisfies the optimality condition for the stationary model as follows:

$$\begin{aligned}
\frac{\delta L(q)}{\delta q(\mathbf{x})} &= \int_{\Omega} \left[\frac{\partial L(q(\mathbf{k}), \mathbf{k})}{\partial q(\mathbf{k})} \frac{\delta q(\mathbf{k})}{\delta q(\mathbf{x})} f^q_{\mathbf{X}}(\mathbf{k}) + L(q(\mathbf{k}), \mathbf{k}) \frac{\partial f^q_{\mathbf{X}}(\mathbf{k})}{\partial q(\mathbf{x})} \right] d\mathbf{k} \\
&= \int_{\Omega} \left[\frac{\partial L(q(\mathbf{k}), \mathbf{k})}{\partial q(\mathbf{k})} \delta(\mathbf{k} - \mathbf{x}) f^q_{\mathbf{X}}(\mathbf{k}) + L(q(\mathbf{k}), \mathbf{k}) \frac{\partial f^q_{\mathbf{X}}(\mathbf{k})}{\partial q(\mathbf{x})} \right] d\mathbf{k} \\
&= \int_{\Omega} \frac{\partial L(q(\mathbf{k}), \mathbf{k})}{\partial q(\mathbf{k})} f^q_{\mathbf{X}}(\mathbf{k}) \delta(\mathbf{k} - \mathbf{x}) d\mathbf{k} + \int_{\Omega} L(q(\mathbf{k}), \mathbf{k}) \frac{\partial f^q_{\mathbf{X}}(\mathbf{k})}{\partial q(\mathbf{x})} d\mathbf{k} \\
&= \frac{\partial L(q(\mathbf{x}), \mathbf{x})}{\partial q(\mathbf{x})} f^q_{\mathbf{X}}(\mathbf{x}) + \int_{\Omega} L(q(\mathbf{k}), \mathbf{k}) \frac{\partial f^q_{\mathbf{X}}(\mathbf{k})}{\partial q(\mathbf{x})} d\mathbf{k} = 0, \quad \forall \mathbf{x} \in \Omega,
\end{aligned} \tag{2}$$

where we apply the *chain rule* first and then the *product rule* of the functional derivative [Appendix A of 20] to obtain the first line, replace $\delta q(\mathbf{k})/\delta q(\mathbf{x})$ with the Dirac delta function $\delta(\mathbf{k} - \mathbf{x})$ to obtain the second line, and apply its sifting property $g(\mathbf{x}) = \int_{\Omega} g(\mathbf{k}) \delta(\mathbf{k} - \mathbf{x}) d\mathbf{k}$ for every continuous function $g(\cdot)$ [21] to obtain the fourth line. The derivation of Eq. (2) is motivated by the Kohn–Sham approach to reduce the dimensionality of multi-body problems in Physics [19].

The optimality condition (2) for the stationary model has two components. When the order-up-to level (i.e., the policy) changes from $q(\mathbf{x})$ to $q(\mathbf{x}) + \delta q(\mathbf{x})$ for the initial inventory $\mathbf{x}$:

- The first term is the contribution of this policy change to the average total cost $L(q)$, assuming that the initial inventory distribution remains the same.
- The second term is the contribution of the policy change to the average total cost $L(q)$ due to the *change in the initial inventory distribution*. This second term, which we refer to as *externality*, captures *the long-term impact of ordering decisions* since an equilibrium inventory distribution is reached only after infinitely many periods.

The externality term is the main source of complexity in the exact optimality condition (2). Specifically, it is difficult to evaluate the derivative $\partial f^q_{\mathbf{X}}(\mathbf{k})/\partial q(\mathbf{x})$ representing the impact of the change in the policy $q(\mathbf{x})$ on the distribution of the initial inventory vector.

To resolve this complexity, we can approximate the externality term using any simple and reasonably good policy $\tilde{q}$. Specifically, we replace $\partial f^q_{\mathbf{X}}(\mathbf{k})/\partial q(\mathbf{x})$ with $\partial f^{\tilde{q}}_{\mathbf{X}}(\mathbf{k})/\partial \tilde{q}(\mathbf{x})$ along with some necessary modifications due to normalization. Adopting the idea of the local density approximation [19], we utilize the CBS policy, which is simple to optimize and known to be a reasonably good policy for many inventory models. When switching the policy from the optimal state-dependent policy q^* to the optimal CBS policy q^*_c, we use the following two approximations:

1. $L(q^*(\mathbf{x}), \mathbf{x}) \delta f^{q^*}_{\mathbf{X}}(\mathbf{x}) \approx L(q^*_c, \mathbf{x}) \delta f^{q^*_c}_{\mathbf{X}}(\mathbf{x})$: This approximation implies that the expected change in the one-period cost (originating from the change of the initial inventory distribution $\delta f^q_{\mathbf{X}}(\mathbf{x}) = f^{q+\delta q}_{\mathbf{X}}(\mathbf{x}) - f^q_{\mathbf{X}}(\mathbf{x})$) is maintained when switching from q^* to q^*_c.
2. $\delta q^*(\mathbf{x}) f^{q^*}_{\mathbf{X}}(\mathbf{x}) \approx \delta q^*_c$ (or equivalently, $\partial q_c / \partial q(\mathbf{x})\big|_{q=q^*} \approx f^{q^*}_{\mathbf{X}}(\mathbf{x})$): This approximation implies that the change of the order amount $(\delta q^*(\mathbf{x}))$ at $\mathbf{x}$ is weighted by its probability density $f^{q^*}_{\mathbf{X}}(\mathbf{x})$ when switching from q^* to q^*_c.

[1] If $\mathbf{X}$ is a mixture of continuous and discrete random vectors, we should use either the Dirac delta function or probability mass function. We omit this discussion since the analysis is essentially the same.

Following these approximations, we simplify the externality term in Eq. (2) as we present in Eq. (3), in which $V_{ex}(q_c)$ follows Eq. (4):

$$\int_{\Omega} L(q^*(\mathbf{k}),\mathbf{k}) \left.\frac{\partial f_{\mathbf{X}}^{q}(\mathbf{k})}{\partial q(\mathbf{x})}\right|_{q=q^*} d\mathbf{k} \approx \int_{\Omega} L(q_c^*(\mathbf{k}),\mathbf{k}) \left.\frac{\partial f_{\mathbf{X}}^{q_c}(\mathbf{k})}{\partial q_c}\right|_{q_c=q_c^*} \left.\frac{\partial q_c}{\partial q(\mathbf{x})}\right|_{q=q^*} d\mathbf{k} \approx V_{ex}(q_c^*) f_{\mathbf{X}}^{q^*}(\mathbf{x}), \tag{3}$$

where

$$V_{ex}(q_c) = \int_{\Omega} L(q_c,\mathbf{k}) \frac{\partial f_{\mathbf{X}}^{q_c}(\mathbf{k})}{\partial q_c} d\mathbf{k}. \tag{4}$$

Combining Eqs. (2) and (3), we derive the approximate optimality condition (5) for the stationary model, conditioned on $\mathbf{x}$ being recurrent (i.e., $f_{\mathbf{X}}^{q}(\mathbf{x}) > 0$):

$$\left(\frac{\partial L(q,\mathbf{x})}{\partial q(\mathbf{x})} + V_{ex}(q_c^*)\right) f_{\mathbf{X}}^{q}(\mathbf{x}) = 0, \qquad \forall \mathbf{x} \in \Omega \implies \frac{\partial L(q,\mathbf{x})}{\partial q(\mathbf{x})} + V_{ex}(q_c^*) = 0. \tag{5}$$

Similar to Eq. (2), the optimality condition (5) has two components, which we refer to as the *marginal internal cost* (MIC) and the *marginal external cost* (MEC):

- MIC is the first term $\partial L(q,\mathbf{x})/\partial q(\mathbf{x})$, and
- MEC is the second term $V_{ex}(q_c^*)$, which is a constant since it is independent of the initial inventory $\mathbf{x}$ under the CBS approximation.

Without the MEC term, Eq. (5) reduces to the optimality condition of a standard single-decision inventory model, which is easy to solve. But the MEC term does not increase the computational complexity of solving Eq. (5) as it is simply a constant. Nevertheless, it plays an important role in minimizing the average cost. By solving Eq. (5), we obtain the approximate optimal policy q_h^*, which is a state-dependent policy (due to the MIC term) as the optimal policy.

3. Applying the framework for a perishable inventory model

This section showcases how our framework, described in Section 2, can be used to analyze the classic model by Nahmias [3]. The four cost parameters include: purchase ($c \geq 0$ per unit), holding ($h \geq 0$ per period per unit), shortage (lost revenue) ($r \geq 0$ per unit), and wastage ($\theta \geq 0$ per unit), which are charged linearly against ordering, holding, unsatisfied demand, and wastage, respectively. To avoid trivial cases, we only consider the case with $r - c > 0$ and $\theta + c > 0$. Holding cost applies to the remaining units at the end of each period, including those that become outdated. Wastage cost applies to the units not used by the end of their m-period lifetime. As in Section 2, we assume:

- Periodic-review with lost sales,
- Fixed product lifetime,
- No lead time,
- Independent and identically distributed (i.i.d.) demand (denoted by D),
- FIFO issuing policy,
- No fixed ordering cost.

This section describes how to derive the approximate optimality condition (5) for this model, and then in Section 4, we compare the total costs under the approximate optimal policy and optimal policy obtained using the DP formulation.

To derive the one-period cost $L(q,\mathbf{x})$, we evaluate the costs associated with a single ordering decision at the beginning of period 1, considering that the holding and shortage costs are incurred in period 1 and wastage cost is incurred in period m. Let the random variable $A_i(\mathbf{x}^{i-1})$ represent the i-period *effective demand*, i.e., the total demand and wastage from periods 1 to i (excluding the wastage in period i). Denoting the random variables for demand and wastage in period i by D_i and R_i, respectively, $A_i(\mathbf{x}^{i-1})$ follows:

$$A_i(\mathbf{x}^{i-1}) = \sum_{j=1}^{i} D_j + \sum_{j=1}^{i-1} R_j, \quad i = 1\ldots,m, \tag{6}$$

and can be formulated using the following recursive representation:

$$A_i(\mathbf{x}^{i-1}) = \begin{cases} D_1 & \text{if } i = 1, \\ A_{i-1}(\mathbf{x}^{i-2}) + R_{i-1} + D_i & \text{if } i = 2,\ldots,m, \end{cases} \tag{7}$$

where

$$R_{i-1} = \begin{cases} 0 & \text{if } i = 1, \\ [w_{i-1} - A_{i-1}(\mathbf{x}^{i-2})]^+ & \text{if } i = 2,\ldots,m. \end{cases} \tag{8}$$

According to the notation conventions introduced earlier, $A_m(\mathbf{x}^{m-1}) = A_m(\mathbf{x})$ and $q = q(\mathbf{x})$. We can express the one-period profit $\Pi(q,\mathbf{x})$ by considering the expected sales revenue (generated in period 1), expected holding cost (incurred in period 1), and the expected wastage cost (incurred in period m):

$$\begin{aligned}\Pi(q,\mathbf{x}) &= (r-c)\mathbb{E}_D[D - (D-q)^+] - h\mathbb{E}_D[q-D]^+ \\ &\quad - (\theta + c)\mathbb{E}_{A_m(\mathbf{x})}[q - A_m(\mathbf{x})]^+ \\ &= (r-c)\mathbb{E}_D[D] - L(q,\mathbf{x}),\end{aligned} \tag{9}$$

where the one-period cost $L(q,\mathbf{x})$ follows Eq. (10):[2]

$$L(q,\mathbf{x}) = h\mathbb{E}_D[q-D]^+ + (r-c)\mathbb{E}_D[D-q]^+ + (\theta+c)\mathbb{E}_{A_m(\mathbf{x})}[q-A_m(\mathbf{x})]^+, \tag{10}$$

where $[q-D]^+$, $[D-q]^+$, and $[q - A_m(\mathbf{x})]^+$ represent the number of units being held, in shortage, and wasted under policy q, respectively. Therefore, the one-period cost $L(q,\mathbf{x})$ includes three single-period costs: single-period costs of units holding (incurred in period 1), in shortage (incurred in period 1), and wasted (incurred in period m). Alternatively, we could include $c\mathbb{E}_D[D]$ (from Eq. (9)) in the $L(q,x)$ expression; however, since $c\mathbb{E}_D[D]$ is constant (i.e., independent of q and $\mathbf{x}$), including or not including $c\mathbb{E}_D[D]$ in $L(q,x)$ does not affect the analysis.

Following Eq. (1), we can represent the average total cost $L(q)$ as follows:

$$L(q) = hn_h(q) + (r-c)n_s(q) + (\theta + c)n_w(q), \tag{11}$$

where the expected number of units being held ($n_h(q)$), in shortage ($n_s(q)$), and wasted ($n_w(q)$) under policy q follow:

$$n_h(q) = \mathbb{E}_{\mathbf{X}}\mathbb{E}_D[q(\mathbf{X}) - D]^+, \tag{12}$$

$$n_s(q) = \mathbb{E}_{\mathbf{X}}\mathbb{E}_D[D - q(\mathbf{X})]^+, \tag{13}$$

$$n_w(q) = \mathbb{E}_{A_m(\mathbf{X})}[q(\mathbf{X}) - A_m(\mathbf{X})]^+. \tag{14}$$

Note that the first term in Eq. (9), $(r-c)\mathbb{E}_D[D]$, is independent of the policy q and the initial inventory $\mathbf{x}$. Thus, the average total profit is maximized when the average total cost $L(q)$ is minimized.

3.1. Marginal Internal Cost (MIC)

To evaluate the MIC term (the first term in the approximate optimality condition (5)), we need $L(q,\mathbf{x})$, which in turn depends on the m-period effective demand $A_m(\mathbf{x})$. Let $F_Y(\cdot)$ and $f_Y(\cdot)$ represent the cumulative distribution function (CDF) and probability density function (PDF) of a random variable Y, respectively. The CDF of the m-period effective demand $A_m(\mathbf{x})$ (i.e., $F_{A_m(\mathbf{x})}(\cdot)$) is obtained by applying a modified convolution formula recursively as shown in Proposition 1. For some distributions, it might be possible to derive the closed-form expression for $F_{A_m(\mathbf{x})}(\cdot)$ (e.g., for an exponential distribution [22]). Otherwise, we can numerically obtain $F_{A_m(\mathbf{x})}(\cdot)$.

[2] For the ease of exposition, we incorporate the purchase cost when the unit is either sold or perished; i.e., $r - c > 0$ and $\theta + c > 0$ represent the shortage (understocking) and the wastage (overstocking) costs.

Proposition 1. *The CDF of $A_m(\mathbf{x})$ is obtained by applying the following recursively:*

$$F_{A_{i+1}(\mathbf{x}^i)}(z) = \begin{cases} \int_{\xi=0}^{z-x_i} F_{A_i(\mathbf{x}^{i-1})}(z-\xi) f_{D_{i+1}}(\xi) d\xi & \text{if } z > x_i, \quad i = 1,\dots,m-1, \\ 0 & \text{if } z \le x_i, \quad i = 1,\dots,m-1, \\ F_D(z) & \text{if } i = 0. \end{cases} \tag{15}$$

We can now represent $L(q,\mathbf{x})$ and its partial derivative (i.e., MIC) as Eqs. (16) and (17), respectively:

$$L(q,\mathbf{x}) = h\int_0^q (q-z) f_D(z)dz + (r-c)\int_q^\infty (z-q) f_D(z)dz$$
$$+(\theta+c)\int_0^q (q-z) f_{A_m(\mathbf{x})}(z)dz. \tag{16}$$

$$MIC : \frac{\partial L(q,\mathbf{x})}{\partial q(\mathbf{x})} = -(h+r-c)\bar{F}_D(q) + (\theta+c)F_{A_m(\mathbf{x})}(q) + h, \tag{17}$$

where $\bar{F}_D(q) = 1 - F_D(q)$ is the complementary CDF of the demand distribution. Furthermore, we can confirm that $L(q,\mathbf{x})$ is strictly convex because:

$$\frac{\partial^2 L(q,\mathbf{x})}{\partial q(\mathbf{x})^2} = (h+r-c)f_D(q) + (\theta+c)f_{A_m(\mathbf{x})}(q) > 0, \qquad \forall q \in [0,\infty). \tag{18}$$

3.2. Marginal External Cost (MEC)

The MEC term (i.e., $V_{ex}(q_c)$ in the approximate optimality condition (5)) is fundamental in our method. As the policy changes, the initial inventory distribution also changes. The marginal internal cost MIC term does not capture this effect since it assumes that the distribution remains unchanged. Through the marginal external cost MEC term, we capture the impact of the policy change on the average total cost $L(q)$ due to the change in the initial inventory distribution. In other words, the MIC term captures the short-term impact of ordering decisions while the MEC term captures the l*ong-term* impact of ordering decisions.

To evaluate the MEC term, we substitute the expression for $L(q_c,\mathbf{x})$ from Eq. (16) into Eq. (4). The first and second terms in Eq. (16) do not contribute to the MEC term as they do not depend on $\mathbf{x}$ because:

$$\int_\Omega \frac{\partial f_{\mathbf{X}}^{q_c}(\mathbf{x})}{\partial q_c} d\mathbf{x} = \frac{\partial[\int_\Omega f_{\mathbf{X}}^{q_c}(\mathbf{x})d\mathbf{x}]}{\partial q_c} = \frac{\partial 1}{\partial q_c} = 0, \tag{19}$$

where we use the Leibniz's rule [see, e.g., §2.4 of 23] to interchange integration and differentiation since the integration of the derivative of $f_{\mathbf{X}}^{q_c}(\cdot)$ over a bounded region is finite (note: $f_{\mathbf{X}}^{q_c}(\cdot)$ has a bounded support under the CBS policy q_c). Therefore, the only cost term in Eq. (16) contributing to the MEC term $V_{ex}(q_c)$ is the wastage cost term. As a result, by substituting Eq. (16) into Eq. (4), we derive:

$$MEC : V_{ex}(q_c) = (\theta+c)w_{ex}(q_c), \tag{20}$$

where

$$w_{ex}(q_c) = \int_\Omega \int_0^{q_c} (q_c - z) f_{A_m(\mathbf{k})}(z)dz \frac{\partial f_{\mathbf{X}}^{q_c}(\mathbf{k})}{\partial q_c} d\mathbf{k}. \tag{21}$$

Let $\mathbf{X}^{q_c}$ denote the initial inventory random vector under the CBS policy q_c. We can compute $w_{ex}(q_c)$ following Eq. (22) by discretizing q_c with step size Δ and evaluating the difference between two expectations:

$$w_{ex}(q_c) = \frac{1}{\Delta}\left(n_w^\Delta(q_c) - n_w(q_c)\right), \tag{22}$$

where $n_w(q_c)$ (the expected wastage under the CBS policy q_c) follows from Eq. (14) and $n_w^\Delta(q_c)$ follows the following equation:

$$n_w^\Delta(q_c) = \mathbb{E}[q_c - A_m(\mathbf{X}^{q_c+\Delta})]^+. \tag{23}$$

By replacing the MIC term Eq. (17) and the MEC term Eq. (20) in the approximate optimality condition (5), we obtain the approximate optimality condition as follows:

$$(\theta+c)\left(F_{A_m(\mathbf{x})}(q) + w_{ex}(q_c^*)\right) + hF_D(q) = (r-c)\bar{F}_D(q). \tag{24}$$

Eq. (24) is the optimality condition to find our approximate optimal order-up-to level policy $q_h^*(\mathbf{x})$, which can be obtained following marginal analysis. The left hand side of Eq. (24) represents the *marginal cost*, i.e., the additional wastage and holding costs incurred due to a unit increase in the order-up-to level q given the initial inventory $\mathbf{x}$. The right hand side of Eq. (24) represents the *marginal benefit*, i.e., the additional benefit from a reduction in the shortage cost due to a unit increase in q. The approximate optimality condition (24) is almost equivalent to the optimality condition for the newsvendor model except that it incorporates the marginal externality term $w_{ex}(q_c^*)$ that captures the long term impact of policy change on the average total cost.

For simplicity of the discussions, we rearrange the approximate optimality condition (24) by moving all the terms that are independent of the initial inventory $\mathbf{x}$ to the right side of the equation as follows:

$$(\theta+c)F_{A_m(\mathbf{x})}(q) = (h+r-c)\bar{F}_D(q) - h - (\theta+c)w_{ex}(q_c^*). \tag{25}$$

Next, we discuss the analytical properties of the approximate optimal policy q_h^* resulted from solving Eq. (25).

3.3. Analytical properties of the approximate optimal policy

This section discusses the properties of the approximate optimal policy q_h^* satisfying Eq. (25). We also assume $f_D(d)$ is finite for all positive d and has unbounded support, which guarantees that $F_D(d)$ is continuous for $d \in [0,+\infty)$ and $\mathbf{x} = 0$ is always recurrent.

3.3.1. Existence of the unique approximate optimal policy

The externality component $w_{ex}(q_c^*)$ in the approximate optimality condition (25) is constant. The right hand side of Eq. (25) is monotonically decreasing in q, while its left hand side is monotonically increasing in q. Therefore, we can easily verify if the unique solution (approximate optimal policy q_h^*) exists by comparing the left and right hand sides of the approximate optimality condition (25) at $q = 0$ and $q \to \infty$. When $w_{ex}(q_c^*)$ is omitted from the equation, the existence of the unique solution is obvious. To discuss the case with $w_{ex}(q_c^*)$, the following proposition is useful.

Proposition 2. *The externality component in the approximate optimality condition* (25) *is non-positive and bounded; i.e., $-1 < w_{ex}(q_c) \le 0, \forall q_c \ge 0$.*

The result derived in Proposition 2 allows us to prove the existence of a unique solution for Eq. (25) (the approximate optimal policy q_h^*), as we prove in Proposition 3.

Proposition 3. *There exists a unique finite order-up-to level (approximate optimal policy) $q_h^*(\mathbf{x})$ satisfying the optimality condition* (25) *for any initial inventory vector $\mathbf{x} \in \Omega$.*

Although the exact analysis of $w_{ex}(q_c^*)$ is formidable, the numerical evaluation of $w_{ex}(q_c^*)$ (following the CBS policy) is simple. We can thus find the unique approximate optimal policy $q_h^*(\mathbf{x}), \forall \mathbf{x} \in \Omega$, numerically.

3.3.2. Approximate optimal policy vs. CBS

This section discusses conditions under which the approximate optimality policy approaches a simple and easy-to-manage state-independent (i.e., CBS) policy. Proposition 4 shows the necessary and sufficient condition for such cases. For simplicity, let $A_m = A_m(\mathbf{0}) = \sum_{j=1}^m D_j$, which is the sum of m i.i.d. demand random variables; thus, the probability distribution of A_m is the convolution of m demand distributions.

Proposition 4. *The approximate optimal policy approaches CBS if and only if the marginal wastage (the extra wastage due to an increase in the* *order-up-to level) at $q_h^*(\mathbf{0})$ approaches zero, i.e., $F_{A_m}(q_h^*(\mathbf{0})) = \Pr(A_m \leq q_h^*(\mathbf{0})) \to 0$.*

When q_h^* approaches CBS, we can find the approximate optimal policy analytically as specified in Corollary 1.

Corollary 1. *When $F_{A_m}(q_h^*(\mathbf{0})) \to 0$ (i.e., q_h^* approaches the optimal CBS according to Proposition 4), both the wastage and externality terms approach zero (i.e., $n_w(q_h^*), w_{ex}(q_c^*) \to 0$), and consequently the approximate optimal policy follows:*

$$q_h^* \to q_c^* \ = \ F_D^{-1}(\gamma^*),$$

where the critical ratio $\gamma^* = \dfrac{r-c}{h+r-c}$.

Note that for the solution q_c^* to be finite, we require $h > 0$ since $f_D(d)$ has an unbounded support. According to Corollary 1, when the system observes no wastage and no externality, the approximate optimal policy q_h^* becomes a function of the critical ratio γ^*, which represents the trade-off between shortage cost $r-c$ (>0) and holding cost h (>0).

Corollary 1 implies other results. For example, a condition that would result in $F_{A_m}(q_h^*(\mathbf{0})) \to 0$, and subsequently the approximate optimal policy approaches the optimal CBS policy, is when the product lifetime grows. As m grows, the impacts of the wastage and the externality diminish. Since the condition $F_{A_m}(q_h^*(\mathbf{0})) \to 0$ is achieved only at the limit of $m \to \infty$, we can prove in Corollary 2 that for a specific parameter setting, and within a margin of error, there is a threshold on m beyond which the approximate optimal policy approaches the optimal CBS policy.

Corollary 2. *The approximate optimal policy q_h^* becomes CBS with the margin of error α when the lifetime $m \geq m_\alpha$, where m_α denotes the least positive integer to ensure $F_{A_m}(F_D^{-1}(\gamma^*)) \leq \alpha$.*

To make Corollary 2 easy to implement in practice, the condition $F_{A_m}(F_D^{-1}(\gamma^*)) \leq \alpha$ may be replaced by $\Phi^m(F_D^{-1}(\gamma^*)) \leq \alpha$ using the normal CDF $\Phi^m(\cdot)$ that approximates the exact CDF (of the convolution of m demands) $F_{A_m}(\cdot)$ following the Central Limit Theorem. Finally, besides the result in Corollary 2, there are other conditions that would lead to $F_{A_m}(F_D^{-1}(\gamma^*)) \leq \alpha$. For example, (1) as h increases or (2) as the demand variability decreases, $F_{A_m}(F_D^{-1}(\gamma^*))$ shrinks, and therefore, the approximate optimal policy approaches the optimal CBS policy. These conditions are consistent with what is also known for the optimal policy [5,7].

In the next section, we evaluate the accuracy of the approximate optimal policy, resulting from our marginal analysis framework, and explain the intuition of our marginal analysis.

4. Numerical validation

In this section, we compare our approximate optimal policy q_h^* and the optimal policy q^* through numerical experiments, and then, we use the experiments to elaborate more on the analytical results on q_h^* derived in the previous section.

4.1. The marginal analysis algorithm

We summarize the steps of our marginal analysis framework to compute the approximate optimal policy q_h^* in Table 2.

We require the pre-processing step only once for each combination of product lifetime m and demand distribution. The evaluation of $n_h(q_c)$, $n_s(q_c)$, $n_w(q_c)$, and $n_w^\Delta(q_c)$ for the entire q_c values in the pre-processing step is computationally fast based on the Monte Carlo simulation (we run a 2,000-period simulation for each value of q_c). The marginal analysis step is efficiently carried out by numerically solving Eq. (25), or performing marginal analysis (by increasing q, the left-hand side of Eq. (25) increases while the right-hand side decreases; q_h^* is the intersecting point of the left and right-hand sides of Eq. (25)).

Table 2
Marginal analysis algorithm.

Pre-processing (performed for each combination of m and demand distribution):
Derive $F_{A_m(\mathbf{x})}(q)$ (Eq. (15)).
Discretize q_c and $\mathbf{x}$ for continuous distributions.
For each q_c, simulate a system with CBS policy and evaluate:
$n_h(q_c)$ (Eq. (12)),
$n_s(q_c)$ (Eq. (13)),
$n_w(q_c)$ (Eq. (14)),
$n_w^\Delta(q_c)$ (Eq. (23)).
Marginal analysis (performed for each combination of c, h, r, and θ):
For each value of q_c, evaluate $L(q_c)$ (Eq. (11)).
Find $q_c^* = arg\,min_{q_c} L(q_c)$.
Evaluate $w_{ex}(q_c^*)$ (Eq. (22)).
For each $\mathbf{x} \in \Omega$, conduct marginal analysis to determine $q_h^*(\mathbf{x})$ (Eq. (25)).
Find the order amount $x_m = [q_h^*(\mathbf{x}) - x]^+$.

As a benchmark policy, we use value iteration algorithm to solve the DP formulation of the problem. We run experiments for the exponential and Poisson demands with mean of 10, and for product lifetimes of $m = 2$ and $m = 3$. We consider smaller m cases because policies are highly state-dependent in such cases—policies approach CBS as m grows large (see Corollary 2). Also, the computation of the optimal policy based on the DP formulation is formidable for larger values of m (especially for the exponential demand distribution). In the experiments, we set $c = 0$ since the optimal policy is the same for any combination of the parameters that result in the same $r-c$ and $\theta + c$. The values of other parameters in the experiments are specified in the first column of Table 3. For the exponential demand case, we discretize the continuous state space with a step size of 0.1, and use a closed-form solution for $F_{A_m(\mathbf{x})}(\cdot)$ [22], while for the Poisson demand case, we recursively obtain $F_{A_m(\mathbf{x})}(\cdot)$ numerically (starting from $F_D(\cdot)$).

Table 3 compares the computation times, in seconds, to find our approximate optimal policy q_h^* using a spreadsheet on a personal computer and the optimal policy q^* by implementing the DP algorithm on a high-performance virtual machine. As the table shows, the computation time for q_h^* is significantly shorter than that of q^* for all of our experiments we tested. While the computation time of q^* varies from one problem to another problem as the parameters change, the computation time for obtaining q_h^* using the marginal analysis algorithm is insensitive to the specific parameters of a problem. The comparison of the two approaches for larger m is not possible as the computation time for the DP algorithm is formidable. However, we know from Corollary 2 that as m grows the optimal policy approaches CBS and therefore the need to use the DP algorithm diminishes, and we can simply use Corollary 1 to obtain the optimal CBS level.

4.2. Evaluations of average costs and order amounts

We compare the average total costs $L(q)$ under our approximate optimal policy q_h^* and the optimal DP policy q^*. To evaluate $L(q)$ (following Eq. (11)), we obtain $n_h(q)$, $n_s(q)$, $n_w(q)$ for q_h^* and q^* using a Monte Carlo simulation; this is necessary because the closed-form expression for the initial inventory distribution $f_{\mathbf{X}}^q(\cdot)$ is unknown. We define the percentage cost deviation: $G = (L_h^* - L^*)/L^*$, where L_h^* and L^* correspond to $L(q)$ following our approximate policy q_h^* and the optimal DP policy q^*, respectively. The results summarized in Table 4 indicate that the performance of our approximate policy q_h^* is very close to that of the DP policy: The average cost deviation from the DP policy is around 0.05% and the maximum cost deviation is 0.34% for all forty experiments we tested.

We also examine the accuracy of our approximate optimal policy q_h^* with respect to individual order amount. Figs. 2(a) and 2(b) show examples of this comparison: We observe that the order amounts following policy q_h^* closely match those from the optimal policy q^*

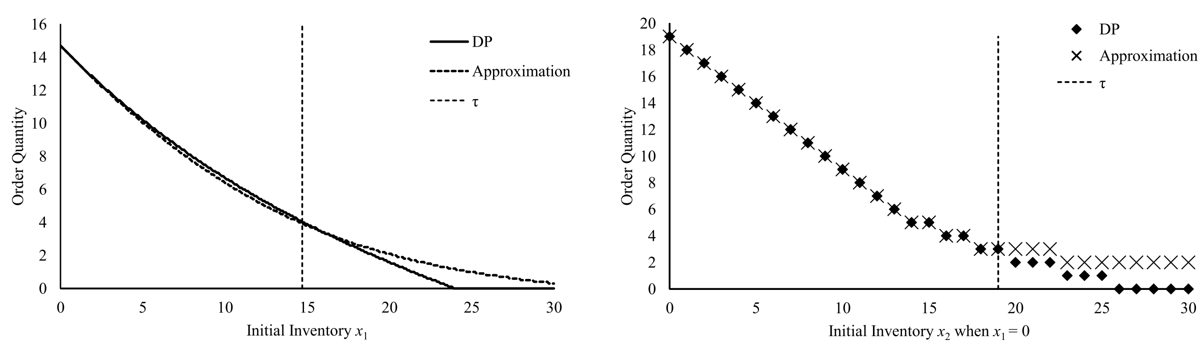
Table 3
Computation times (seconds).

h, r, θ	Exponential Demand				Poisson Demand			
	$m = 2$		$m = 3$		$m = 2$		$m = 3$	
	DP	Approx	DP	Approx	DP	Approx	DP	Approx
0, 5, 5	3087.00	5.02	92820.16	103.55	123.94	4.22	20937.45	5.72
0, 5, 10	1987.83	5.11	60150.88	103.08	116.56	4.10	20222.16	5.71
0, 5, 20	1232.70	5.12	37582.14	102.66	109.30	4.04	20084.26	5.73
0, 8, 7	3357.25	5.09	100010.54	103.07	120.78	4.21	19773.06	5.51
0, 10, 5	4601.25	5.10	135755.33	102.69	130.65	4.09	20632.77	5.60
1, 5, 5	1852.03	5.05	30269.52	103.16	93.20	4.17	19892.16	5.70
1, 5, 10	1321.73	5.05	24330.17	102.73	93.71	4.09	19990.42	5.62
1, 5, 20	895.28	4.98	18399.56	103.00	92.04	4.10	19715.03	5.44
1, 8, 7	2396.87	5.01	46478.69	103.15	100.67	4.23	19717.75	5.64
1, 10, 5	3311.25	5.07	46478.69	102.99	100.74	4.26	19993.63	5.46
Average	2404.32	5.06	60644.00	103.00	109.16	4.15	20095.87	5.61

Table 4
Comparison between the DP and approximate policies.

h, r, θ	Exponential Demand						Poisson Demand					
	$m = 2$			$m = 3$			$m = 2$			$m = 3$		
	DP	$G\%$	MAD	DP	$G\%$	MAD	DP	$G\%$	MAD	DP	$G\%$	MAD
0, 5, 5	19.84	0.04	0.28	12.14	0.34	0.62	1.47	0.14	0.13	0.13	0.27	0.00
0, 5, 10	25.40	0.06	0.26	16.05	0.07	0.26	2.09	0.11	0.20	0.19	0.04	0.11
0, 5, 20	30.74	0.02	0.14	20.24	0.02	0.33	2.92	0.08	0.07	0.26	0.00	0.00
0, 8, 7	30.06	0.05	0.29	18.31	0.20	0.48	2.16	0.00	0.06	0.19	0.00	0.00
0, 10, 5	29.19	0.09	0.38	17.49	0.17	0.46	1.95	0.00	0.06	0.17	0.00	0.00
1, 5, 5	25.39	0.01	0.14	20.88	0.00	0.05	5.26	0.27	0.43	4.93	0.00	0.00
1, 5, 10	28.93	0.01	0.14	22.69	0.00	0.04	5.52	0.00	0.00	4.93	0.00	0.00
1, 5, 20	32.81	0.00	0.05	25.03	0.01	0.10	5.88	0.00	0.00	4.94	0.00	0.00
1, 8, 7	36.51	0.02	0.17	28.38	0.02	0.16	6.36	0.00	0.00	5.68	0.00	0.00
1, 10, 5	38.25	0.02	0.18	30.24	0.04	0.25	6.63	0.00	0.20	6.05	0.00	0.00
Average		0.033	0.202		0.088	0.274		0.060	0.116		0.027	0.011
Maximum		0.09	0.38		0.34	0.62		0.27	0.20		0.27	0.11

Notes: Results are based on 10^6-period Monte Carlo simulations (10^4 burn-in periods). The underlined values specify that the optimal policy is CBS.



(a) Order amount (exponential demand; $h = 1, m = 2$) (b) Order amount (Poisson demand; $h = 0, m = 3$)

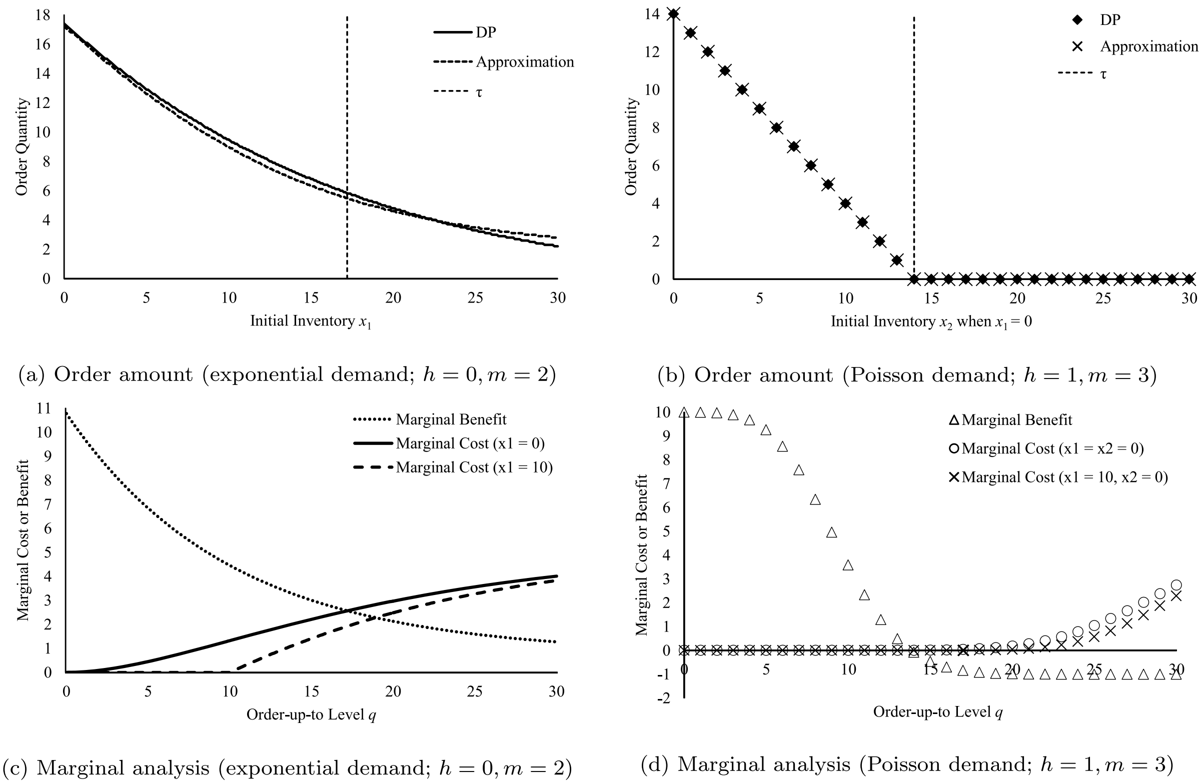
Fig. 2. Comparison between the order amounts under the DP and approximate policies ($r = 10, \theta = 5$). Note: τ separates the recurrent and non-recurrent inventory levels.

in recurrent regions (initial inventory levels observed infinitely many times). The threshold τ for the recurrent region in Figs. 2(a) and 2(b) is specified by the maximum possible initial inventory (i.e., $q_h^*(\mathbf{0})$), as the initial inventory level cannot exceed the order amount at $\mathbf{x} = 0$. The discrepancies in non-recurrent regions do not impact average costs, as these initial inventory levels are transient, and therefore they do not contribute to the average total cost. In Table 4, we report the mean absolute deviation (MAD) between the order amounts of q_h^* and q^* for the recurrent initial inventory levels.[3] The results of small MAD (e.g., on average 0.20 for $m = 2$ and 0.27 for $m = 3$ exponential demand cases) imply that our approximation captures (at least some of) the fundamental structure of the state-dependent optimal policy. In the next subsection, we explain analytical insights into our approximate optimal policy.

4.3. Explaining analytical insights

To illustrate the intuition behind Proposition 4, we compare two contrasting experiments, Figs. 3(a) (exponential demand; $h = 0, m = 2$) and 3(b) (Poisson demand; $h = 1, m = 3$), for which we show the marginal analysis plots in Figs. 3(c) and 3(d), respectively. Fig. 3(c) is a typical plot for the marginal analysis of a *state-dependent* ordering policy, whereas Fig. 3(d) is a typical plot for the marginal analysis of the CBS policy. Each of marginal analysis Figs. 3(c) and 3(d) contains three plots based on Eq. (25): The two non-decreasing lines represent

[3] For $m = 3$, we compare the order amounts when $x_1 = 0$ and $x_2 \geq 0$.



(a) Order amount (exponential demand; $h = 0, m = 2$)

(b) Order amount (Poisson demand; $h = 1, m = 3$)

(c) Marginal analysis (exponential demand; $h = 0, m = 2$)

(d) Marginal analysis (Poisson demand; $h = 1, m = 3$)

Fig. 3. Comparison between the approximate state-dependent and CBS policies ($r = 10, \theta = 5, c = 0$).

Marginal Cost (MC; the left-hand side of Eq. (25) representing the marginal wastage cost) with two different initial inventory levels, while the non-increasing line represents Marginal Benefit (MB; the right-hand side of Eq. (25)). The key is the position of the intersection: $q_h^*(\mathbf{x})$ (the horizontal axis) and $(\theta+c)F_{A_m(\mathbf{x})}(q_h^*(\mathbf{x}))$ (the vertical axis). Proposition 4 tells us that, in an asymptotic manner, the intersecting point for the $\mathbf{x} = \mathbf{0}$ case should lie on the horizontal axis to make the policy CBS. We confirm Proposition 4 holds in our results: If MC and MB intersect above the horizontal axis, then the "optimal" marginal wastage depends on $\mathbf{x}$, making the policy state-dependent (Fig. 3(c)), while if they intersect on the horizontal axis, then the marginal wastage becomes negligible and independent from $\mathbf{x}$, making the policy CBS (Fig. 3(d)). We also confirm that Corollary 1 holds in Fig. 3(d): The approximate optimal policy q_h^* becomes the CBS policy $q_c^* = F_D^{-1}(\gamma^*) = 14$, where $\gamma^* = (r-c)/(h+r-c) = 10/11$.

We now know that policy q_h^* becomes CBS when the marginal wastage is negligible, but when does this occur in practice? Corollary 2 is useful to determine the minimum lifetime m_α to make the policy CBS (within a margin of error α). Consider the $(h, r, \theta, c) = (1, 10, 5, 0)$ case and use $\alpha = 0.01$. If demand D is Poisson, we obtain a relatively small lifetime $m_\alpha = 3$ following Corollary 2 (using the Poisson CDF $F_{A_m}(\cdot)$ or its approximation, the normal CDF $\Phi^m(\cdot)$) to have a CBS policy ($q_c^* = 14$). In contrast, if D is exponential, we require a much larger lifetime, either $m_\alpha = 8$ (using the Erlang CDF $F_{A_m}(\cdot)$) or $m_\alpha = 10$ (using the normal CDF $\Phi^m(\cdot)$), to obtain a CBS policy ($q_c^* = 24$); when $m = 10$, the intersecting point of MC and MB for $\mathbf{x} = 0$ lies on the horizontal axis (Fig. 4(b)) and the policy becomes CBS (Fig. 4(a)).

5. Discussions: Theoretical insights and managerial implications

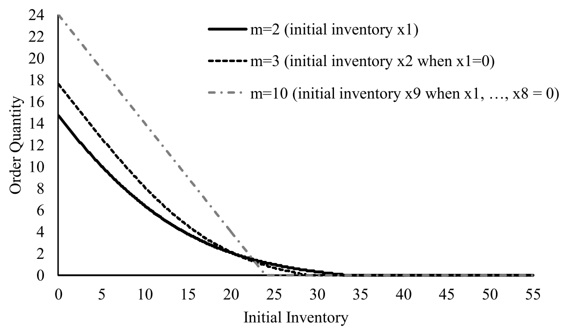
We introduced a framework based on marginal analysis that provides a near-optimal ordering policy (given any initial inventory) in a periodic-review perishable inventory system. Our method is potentially helpful for practitioners who manage perishable inventory systems with non-trivial lifetimes; these systems are complicated to optimize due to their high-dimensional state space [1]. We explain the insights and implications of our method in this section.

5.1. Theoretical insights of the marginal analysis framework

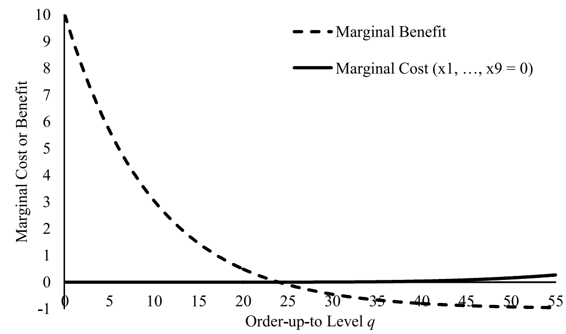
As mentioned above, it is widely known that perishable inventory systems with multi-period lifetime are difficult to solve; this is due to the need to evaluate the compounding impacts of many different decisions over indefinitely many periods. Our approximation resolves this difficulty by establishing a single-decision model with a correction term, referred to as the marginal external cost term, representing the impact of the ordering decision on the long-term inventory distribution. This idea is the same as the *externality* concept in economics [24]. Our approximation also inherits ideas from physics: energy levels of electronic systems are usually modeled by a one-body approximation with an exchange–correlation term [19], which corresponds to the externality term of our method. Both economists and physicists solve extremely complicated high-dimensional problems using a drastically simple one-body model with an additional term representing the correction to the one-body approximation. This idea also seems to work exceptionally well for the perishable inventory examples we have tested in this paper. As we observe many successes in various fields, we believe that this externality concept provides a potentially powerful tool to solve various complex operations management problems.

5.2. Benefit to practitioners

Our method also provides benefits to practitioners managing perishable products with shorter lifetimes. Some short-life product examples include packaged salads, baked goods (e.g., bagels, breads), fresh fruits (e.g., strawberries, blueberries), and deli items (e.g., cold cut meats). These products are wasted on a large scale every day, a few days after their preparation, if they are unsold [25]. For researchers, the exact DP solution of the ordering policy is available to minimize the waste, but its implementation is not simple for practitioners. In contrast,



(a) Order Amount (exponential demand; $m = 2, 3, 10$)



(b) Marginal Analysis (exponential demand; $m = 10$)

Fig. 4. Order amount and marginal analysis ($h = 1, r = 10, \theta = 5, c = 0$).

our approximation method can be implemented on a spreadsheet for lifetimes $m = 2$ and 3 with high accuracy, as demonstrated by our examples in this paper. Retail store owners managing perishables may use our model to approximate the optimal order amount every day before making their orders for the next period.

Another important application area is healthcare. Blood platelets have a lifetime of only five days, but since the testing procedure at blood centers takes two days, platelets have only a three-day remaining lifetime when put into use at hospitals [26]. Since blood platelets are expensive (each unit costs more than \$500 in the U.S. [27]), easy-to-implement software that runs on an Excel spreadsheet could be helpful for practitioners working at blood banks. Our marginal analysis framework is also applicable to fashion products with shorter lifetimes; for example, if a review period is two weeks, fashion products with a lifetime of $m = 3$ represent a store policy that all unsold clothes are replaced after six weeks on the shelf.

5.3. Benefits to educators

Currently, business schools teach the topic of perishable inventory management using a newsvendor model; through this model, students understand the logic of marginal analysis. However, this newsvendor model is restricted to the lifetime $m = 1$. The analysis of the model with lifetime $m = 2$ or above requires the DP method, which utilizes a different logic than marginal analysis. Our method is essentially an extension of the newsvendor model ($m = 1$) with an added externality term, which reflects the change of the initial inventory distribution caused by lifetimes of more than one period ($m \geq 2$). Since our method can be implemented on an Excel spreadsheet (when $m = 2$ or 3), students can understand perishable inventory systems more in-depth and may find the marginal analysis concept useful for other applications.

5.4. Recommendations for managers

When managers waste a large amount of perishable products under a periodic-review policy, there is a high chance that they are not implementing an optimal or approximately optimal ordering policy. Our work recommends two potential actions in this situation. One action is to adopt a simple CBS policy; the other is a more complicated state-dependent (initial inventory-dependent) policy. It should be noted that the simple CBS policy may be optimal or close to optimal when the variability of product demand is sufficiently small. Thus, we recommend that managers first check if the optimal policy is CBS using Corollary 2. If the threshold lifetime m_α calculated from Corollary 2 is less than the actual lifetime, they do not need to use a state-dependent policy since the optimal policy is CBS (or close to it). Instead, they need to evaluate the optimal constant base stock level $F_D^{-1}(\gamma^*)$ following Corollary 1, and then to implement the CBS policy. For other cases and when $m = 2, 3$, they can implement our algorithm on a spreadsheet. (When $m \geq 4$ it is tedious to implement our algorithm on a spreadsheet, and running the algorithm with a programming software package is more suitable.)

6. Concluding remarks

We present a new marginal analysis framework for solving a single-product periodic-review perishable inventory model with lost sales. Our analysis considers the externality effect of ordering perishable products. This externality effect is very complicated since it contains all of the "curse of dimensionality" characteristics of perishable inventory systems; consequently, the optimality condition including externality becomes challenging to solve. However, using the properties of the CBS policy, we can convert this complex optimality condition into a simple newsvendor-type approximate optimality condition that can be easily analyzed, for example, by a spreadsheet. Our approximation method performs well: Numerical experiments show that average costs and *individual* order amount for recurrent initial inventory levels closely match the optimal DP results.

This paper analyzes the cases for products with exponential and Poisson demand distributions and lifetimes of $m = 2, 3$. The restriction for the parameter $m \leq 3$ is not due to the limitation of our algorithm; we will explore the extension of the current work to the $m \geq 4$ case in future works.

Our method captures the complex, long-term impact of ordering perishables in a single externality term using the properties of the CBS policy. However, one could choose different approximations if more accuracy is needed or a different model is envisioned. By taking externality into account properly in various inventory models, one should be able to set up a single decision optimality condition that can provide an accurate approximate policy. We believe our method gives practitioners a simple tool to find near-optimal policies as well as provides researchers an alternative framework to solve complex inventory problems affected by the curse of dimensionality; for example, perishable models with substitutable products or more advanced models (like those considered in [28,29]), with appropriate modifications.

CRediT authorship contribution statement

Katsunobu Sasanuma: Conceptualization, Formal analysis, Data curation, Writing – original draft, Writing – review & editing, Preparing response documents to the review team. **Mohammad Delasay:** Formal analysis, Data curation, Numerical computation and validation, Writing – original draft, Writing – review & editing, Preparing response documents to the review team. **Christine Pitocco:** Conceptualization, Investigation, Editing the original manuscript. **Alan Scheller-Wolf:** Methodological validation, Editing the original manuscript. **Thomas Sexton:** Conceptualization formulation, Editing the original manuscript.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Proposition 1

We can rewrite Eq. (8) as

$$R_i = \max\{w_i - A_i(\mathbf{x}^{i-1}), 0\}, \quad i \geq 1,$$

which is equivalent to

$$A_i(\mathbf{x}^{i-1}) + R_i = \max\{A_i(\mathbf{x}^{i-1}), w_i\}, \quad i \geq 1.$$

Therefore, for $i \geq 1$,

$$F_{A_i(\mathbf{x}^{i-1})+R_i}(\zeta) = \begin{cases} F_{A_i(\mathbf{x}^{i-1})}(\zeta) & \text{if } \zeta \geq w_i, \\ 0 & \text{if } \zeta < w_i. \end{cases}$$

Combining this result with Eq. (7), we obtain

$$\begin{aligned} F_{A_{i+1}(\mathbf{x}^i)}(z) &= Pr\{A_{i+1}(\mathbf{x}^i) \leq z\} = Pr\{A_i(\mathbf{x}^{i-1}) + R_i + D_{i+1} \leq z\} \\ &= Pr\{A_i(\mathbf{x}^{i-1}) + R_i \leq z - D_{i+1}\} \\ &= \int_{\xi=-\infty}^{\infty} F_{A_i(\mathbf{x}^{i-1})+R_i}(z-\xi) f_{D_{i+1}}(\xi) d\xi \\ &= \int_{\xi=0}^{z-w_i} F_{A_i(\mathbf{x}^{i-1})}(z-\xi) f_{D_{i+1}}(\xi) d\xi, \quad \text{if } z > w_i, i \geq 1, \end{aligned}$$

and $F_{A_{i+1}(\mathbf{x}^i)}(z) = 0$, if $z \leq w_i, i \geq 1$. □

Appendix B. Deriving Eq. (22)

We discretize the continuous order-up-to level q_c with a step size of Δ.

$$\begin{aligned} w_{ex}(q_c) &= \int_\Omega \int_0^{q_c} (q_c - z) f_{A_m(\mathbf{k})}(z) dz \frac{\partial f_{\mathbf{X}}^{q_c}(\mathbf{k})}{\partial q_c} d\mathbf{k} \\ &= \int_\Omega \int_0^{q_c} (q_c - z) f_{A_m(\mathbf{k})}(z) dz \frac{f_{\mathbf{X}}^{q_c+\Delta}(\mathbf{k}) - f_{\mathbf{X}}^{q_c}(\mathbf{k})}{\Delta} d\mathbf{k} \\ &= \frac{\int_\Omega \int_0^{q_c} (q_c - z) f_{A_m(\mathbf{k})}(z) dz f_{\mathbf{X}}^{q_c+\Delta}(\mathbf{k}) d\mathbf{k} - \int_\Omega \int_0^{q_c} (q_c - z) f_{A_m(\mathbf{k})}(z) dz f_{\mathbf{X}}^{q_c}(\mathbf{k}) d\mathbf{k}}{\Delta} \\ &= \frac{\mathbb{E}[q_c - A_m(\mathbf{X}^{q_c+\Delta})]^+ - \mathbb{E}[q_c - A_m(\mathbf{X}^{q_c})]^+}{\Delta} \\ &= \frac{n_w^\Delta(q_c) - n_w(q_c)}{\Delta}. \end{aligned}$$

Appendix C. Proof of Proposition 2

We assume (as in Section 3) that $h \geq 0$, $r - c > 0$, $\theta + c > 0$, and $f_D(d) > 0, \forall d \geq 0$. Since we discuss the properties of random initial inventory vectors, it is convenient to use the concept of the first-order stochastic dominance (FSD), which is defined as follows:

Definition 1. A random variable X first-order stochastically dominates another random variable Y ($X \succeq_{FSD} Y$) if and only if $F_X(x) \leq F_Y(x)$, $\forall x \in \mathbb{R}$.

For notational convenience, we write $\mathbf{X} \succeq_{FSD} \mathbf{Y}$ for random vectors $\mathbf{X}$ and $\mathbf{Y}$ if the FSD property holds component-wise: $X_i \succeq_{FSD} Y_i$ for all ith elements of $\mathbf{X}$ and $\mathbf{Y}$. To prove the FSD property for random variables, the following property is convenient and well-known [see, e.g., 30].

Property 1. $X \succeq_{FSD} Y \iff \mathbb{E}_X[f(X)] \geq \mathbb{E}_Y[f(Y)]$ *for any non-decreasing function* $f(\cdot)$.

Next, we present two useful FSD relationships for $A_m(\cdot)$. Let $\mathbf{x}$ and $\mathbf{y}$ be initial inventory vectors in $\Omega = \mathbb{R}_{\geq 0}^{m-1}$, and $\mathbf{X}$ and $\mathbf{Y}$ be the corresponding random vectors.

Lemma 1. $\mathbf{x} \geq \mathbf{y}$ *component-wise* $\implies A_m(\mathbf{x}) \succeq_{FSD} A_m(\mathbf{y})$.

Proof of Lemma 1. The proof follows from Proposition 1 and Definition 1. □

Lemma 2. $\mathbf{X} \succeq_{FSD} \mathbf{Y} \implies A_m(\mathbf{X}) \succeq_{FSD} A_m(\mathbf{Y})$.

Proof of Lemma 2. Combining Property 1 and Lemma 1, we have $\mathbf{x} \geq \mathbf{y}$ component-wise $\implies \mathbb{E}_{A_m(\mathbf{x})}[f(A_m(\mathbf{x}))] \geq \mathbb{E}_{A_m(\mathbf{y})}[f(A_m(\mathbf{y}))]$ for any non-decreasing function $f(\cdot)$. This result indicates that $g(\mathbf{x}) \doteq \mathbb{E}_{A_m(\mathbf{x})}[f(A_m(\mathbf{x}))] = \mathbb{E}_{A_m(\mathbf{X})|\mathbf{X}=\mathbf{x}}[f(A_m(\mathbf{X}))|\mathbf{X} = \mathbf{x}]$ is a non-decreasing function in $\mathbf{x}$ component-wise (because $g(\mathbf{x}) \geq g(\mathbf{y})$ whenever $\mathbf{x} \geq \mathbf{y}$ component-wise). Using Property 1 once again this time with $g(\mathbf{x})$ we define above and the law of total expectation, $\mathbf{X} \succeq_{FSD} \mathbf{Y} \implies \mathbb{E}_{\mathbf{X}}[g(\mathbf{X})] \geq \mathbb{E}_{\mathbf{Y}}[g(\mathbf{Y})] \iff \mathbb{E}_{A_m(\mathbf{X})}[f(A_m(\mathbf{X}))] \geq \mathbb{E}_{A_m(\mathbf{Y})}[f(A_m)(\mathbf{Y})]$ for any non-decreasing function $f(\cdot)$, which indicates $A_m(\mathbf{X}) \succeq_{FSD} A_m(\mathbf{Y})$. □

Let $\mathbf{X}^{q_c}$ and $X_m^{q_c}$ be the initial inventory random vector and the new order under the CBS policy q_c, respectively. Let $\tilde{\mathbf{X}}^{q_c} = (\mathbf{X}^{q_c}, X_m^{q_c}) \in \mathbb{R}_{\geq 0}^m$. Since the entire inventory follows the CBS policy q_c, $\sum_{i=1}^m X_i^{q_c} = q_c$ must hold. Consider increasing the order-up-to level q_c by a positive infinitesimal δ_{q_c}. Then the stationary distribution of the entire inventory (including the new order) shifts from $\tilde{\mathbf{X}}^{q_c}$ to $\tilde{\mathbf{X}}^{q_c+\delta_{q_c}}$. The following relationship holds:

Lemma 3. $\mathbf{X}^{q_c+\delta_{q_c}} \succeq_{FSD} \mathbf{X}^{q_c}$.

Proof of Lemma 3. Define a discrete time stochastic process $\{\tilde{\mathbf{X}}^{q_c}(t), t = 0, 1, 2, \ldots\}$ to represent the entire inventory at time period $t \in \mathbb{Z}_{\geq 0}$. Consider a sample path $\tilde{\mathbf{X}}^{q_c}(t; \omega)$. Without loss of generality, we assume $\mathbf{X}^{q_c}(0; \omega) = \mathbf{0}$ and $X_m^{q_c}(0; \omega) = q_c$, which repeatedly appear one period after we encounter a shortage of inventory (note: $\mathbf{x} = \mathbf{0}$ is recurrent). Suppose the CBS policy is modified from q_c to $q_c + \delta_{q_c}$, where δ_{q_c} is a positive infinitesimal that is non-divisible. Then the sample path at $t = 0$ shifts from $\tilde{\mathbf{X}}^{q_c}(0; \omega) = (\mathbf{0}, q_c)$ to $\tilde{\mathbf{X}}^{q_c+\delta_{q_c}}(0; \omega) = (\mathbf{0}, q_c + \delta_{q_c})$. Assuming that this δ_{q_c} is used last in each age category, either one of the two occurs every period: (1) δ_{q_c} is not used, in which case δ_{q_c} becomes older (or wasted) and shows up in the older age category (or the new order category) in the next period, or (2) δ_{q_c} is used, in which case δ_{q_c} shows up in the same or newer age category in the next period. Hence, the revised sample path is represented as $\tilde{\mathbf{X}}^{q_c+\delta_{q_c}}(t; \omega) = \tilde{\mathbf{X}}^{q_c}(t; \omega) + \delta_{q_c} \mathbf{I}(t; \omega)$, where $\mathbf{I}$ is a random unit vector (one of the age category is 1 and all others are 0) and $\mathbf{I}(t; \omega)$ is its sample path. It follows that, for each age category i, $F_{X_i^{q_c+\delta_{q_c}}}(x) = Pr\{X_i^{q_c+\delta_{q_c}} \leq x\} = Pr\{X_i^{q_c} + \delta_{q_c} I_i \leq x\} \leq Pr\{X_i^{q_c} \leq x\} = F_{X_i^{q_c}}(x)$, $\forall x \in [0, \infty)$. Hence, from Definition 1, we obtain $\tilde{\mathbf{X}}^{q_c+\delta_{q_c}} \succeq_{FSD} \tilde{\mathbf{X}}^{q_c}$, and therefore, $\mathbf{X}^{q_c+\delta_{q_c}} \succeq_{FSD} \mathbf{X}^{q_c}$. (Note: $\tilde{\mathbf{X}}$ and $\mathbf{I}$ are not independent, but the dependency does not affect the conclusion.) □

Lemma 4. $A_m(\mathbf{X}^{q_c+\delta_{q_c}}) \succeq_{FSD} A_m(\mathbf{X}^{q_c})$.

Proof of Lemma 4. The result is immediately obtained from Lemmas 2 and 3. □

Using Property 1 and Lemma 4, we can bound the externality term and obtain Proposition 2.

Proof of Proposition 2. To prove this property, we rewrite the partial derivative with the expression using a positive infinitesimal change δ_{q_c}: $\partial f_{A_m(\mathbf{X}^{q_c})}(z)/\partial q_c = [f_{A_m(\mathbf{X}^{q_c+\delta_{q_c}})}(z) - f_{A_m(\mathbf{X}^{q_c})}(z)]/\delta_{q_c}$.

First part ($w_{ex}(q_c) \le 0$): Changing the order of two integrations and the partial derivative in Eq. (21), we obtain:

$$\begin{aligned} w_{ex}(q_c) &= \int_0^{q_c} (q_c - z)\frac{\partial \int_\Omega f_{A_m(\mathbf{k})}(z) f_{\mathbf{X}}^{q_c}(\mathbf{k})d\mathbf{k}}{\partial q_c} dz \\ &= \int_0^{q_c} (q_c - z)\frac{\partial f_{A_m(\mathbf{X}^{q_c})}(z)}{\partial q_c} dz \\ &= \frac{\int_0^{q_c}(q_c - z) f_{A_m(\mathbf{X}^{q_c+\delta_{q_c}})}(z)dz - \int_0^{q_c}(q_c - z) f_{A_m(\mathbf{X}^{q_c})}(z)dz}{\delta_{q_c}} \\ &= -\left(\frac{-\mathbb{E}[q_c - A_m(\mathbf{X}^{q_c+\delta_{q_c}})]^+ + \mathbb{E}[q_c - A_m(\mathbf{X}^{q_c})]^+}{\delta_{q_c}}\right) \le 0, \end{aligned}$$

where we apply Property 1 and Lemma 4 to $-[q_c - x]^+$, which is a non-decreasing function of x.

Second part ($w_{ex}(q_c) > -1$): Since $[q_c - A_m(\mathbf{x})]^+$ represents the amount of wastage, $\mathbb{E}[q_c - A_m(\mathbf{X}^{q_c})]^+ = \mathbb{E}[X_1^{q_c} - D]^+$ should hold. Also, under the assumption $f_D(d) > 0, \forall d \ge 0$, $F_{A_m(\mathbf{X}^{q_c})}(q_c) < 1$ for a finite q_c. Using these properties, we obtain:

$$\begin{aligned} w_{ex}(q_c) &= \int_0^{q_c} (q_c - z)\frac{\partial f_{A_m(\mathbf{X}^{q_c})}(z)}{\partial q_c} dz \\ &= \frac{\partial \int_0^{q_c}(q_c - z) f_{A_m(\mathbf{X}^{q_c})}(z)dz}{\partial q_c} - \int_0^{q_c} f_{A_m(\mathbf{X}^{q_c})}(z)dz \\ &= \frac{\partial \mathbb{E}[q_c - A_m(\mathbf{X}^{q_c})]^+}{\partial q_c} - F_{A_m(\mathbf{X}^{q_c})}(q_c) \\ &= \frac{\partial \mathbb{E}[X_1^{q_c} - D]^+}{\partial q_c} - F_{A_m(\mathbf{X}^{q_c})}(q_c) \\ &= \frac{\mathbb{E}[X_1^{q_c+\delta_{q_c}} - D]^+ - \mathbb{E}[X_1^{q_c} - D]^+}{\delta_{q_c}} - F_{A_m(\mathbf{X}^{q_c})}(q_c) > -1, \end{aligned}$$

where we apply Property 1 and Lemma 3 to $\mathbb{E}[x - D]^+$, which is a non-decreasing function of x. □

Appendix D. Proof of Proposition 3

We continue to assume (as in Section 3) that $h \ge 0$, $r - c > 0$, $\theta + c > 0$, and $f_D(d) > 0, \forall d \ge 0$. Proposition 3 follows from Proposition 2.

Let $g_{\mathbf{x}}(q) \doteq (\theta+c)F_{A_m(\mathbf{x})}(q) - (h+r-c)\bar{F}_D(q) + h + (\theta+c)w_{ex}(q_c^*)$, where q_c^* is independent of q. Observe that $g_{\mathbf{x}}(q)$ is an increasing function with respect to q as $\partial g_{\mathbf{x}}(q)/\partial q = (\theta+c)f_{A_m(\mathbf{x})}(q) + (h+r-c)f_D(q) > 0$. Also, using Proposition 2, $g_{\mathbf{x}}(0) = -(r-c) + (\theta+c)w_{ex}(q_c^*) < 0$. Finally there always exists a finite $\hat{q}$ satisfying $F_{A_m(\mathbf{x})}(\hat{q}) > 1 - \hat{\delta}$, where $\hat{\delta} = \frac{(\theta+c)(1+w_{ex}(q_c^*))}{h+r+\theta} \in (0,1)$. Such $\hat{q}$ also satisfies $F_D(\hat{q}) > 1 - \hat{\delta}$ (or equivalently, $\bar{F}_D(\hat{q}) < \hat{\delta}$) since $A_m(\mathbf{x}) \ge_{FSD} D$ (see Eq. (7) and Definition 1). Using Proposition 2 and $\hat{\delta}$ defined above, this $\hat{q}$ satisfies

$$\begin{aligned} g_{\mathbf{x}}(\hat{q}) &= (\theta+c)F_{A_m(\mathbf{x})}(\hat{q}) - (h+r-c)\bar{F}_D(\hat{q}) + h + (\theta+c)w_{ex}(q_c^*) \\ &> (\theta+c)(1-\hat{\delta}) - (h+r-c)\hat{\delta} + h + (\theta+c)w_{ex}(q_c^*) \\ &= (\theta+c)(1+w_{ex}(q_c^*)) - (h+\theta+r)\hat{\delta} + h = h \ge 0. \end{aligned}$$

Since $g_{\mathbf{x}}(q)$ is monotonic, we can conclude that there exists a unique and finite solution $q_h^*(\mathbf{x}) \in (0,1)$ that satisfies $g_{\mathbf{x}}(q) = 0$ (and hence the optimality condition (25)) for any initial inventory vector $\mathbf{x} \in \Omega$. □

Finally, we present a corollary of Proposition 3, which is utilized in the proof of Proposition 4.

Corollary 3. *Consider two initial inventory vectors $\mathbf{x_1}, \mathbf{x_2} \in \Omega$, where $\mathbf{x_1} \ne \mathbf{x_2}$. Then*

$$|q_h^*(\mathbf{x_1}) - q_h^*(\mathbf{x_2})| \to 0 \iff |F_{A_m(\mathbf{x_1})}(q_h^*(\mathbf{x_1})) - F_{A_m(\mathbf{x_2})}(q_h^*(\mathbf{x_2}))| \to 0.$$

Proof of Corollary 3. Since the solution to Eq. (25) is unique and finite (Proposition 3), we can write two optimality equations corresponding to initial inventory vectors $\mathbf{x_1}$ and $\mathbf{x_2}$. Note that $w_{ex}(q_c^*)$ in Eq. (25) does not depend on $\mathbf{x}$. By subtracting one from the other, we obtain

$$(\theta+c)(F_{A_m(\mathbf{x_1})}(q_h^*(\mathbf{x_1})) - F_{A_m(\mathbf{x_2})}(q_h^*(\mathbf{x_2}))) = (h+r-c)(\bar{F}_D(q_h^*(\mathbf{x_1})) - \bar{F}_D(q_h^*(\mathbf{x_2}))).$$

The result follows from the assumptions $\theta + c > 0$, $h + r - c > 0$, and continuous $F_D(d)$ for $d \in [0, +\infty)$. □

Appendix E. Proofs of Proposition 4 and Corollaries 1 and 2

We first show the relationship between two solutions with different initial inventory levels (Lemma 5), from which we can determine the upper and lower bounds of the solution (Lemma 6). If the gap between the upper and lower bounds shrinks, a state-dependent policy should approach CBS. The condition to make the gap shrink is provided in Proposition 4.

Lemma 5. *$q_h^*(\mathbf{x_1}) \ge q_h^*(\mathbf{x_2})$ and $F_{A_m(\mathbf{x_1})}(q_h^*(\mathbf{x_1})) \le F_{A_m(\mathbf{x_2})}(q_h^*(\mathbf{x_2}))$ if $\mathbf{x_1} \ge \mathbf{x_2}$ component-wise.*

Proof of Lemma 5. As in the proof of Proposition 3, we define $g_{\mathbf{x}}(q) \doteq (\theta+c)F_{A_m(\mathbf{x})}(q) - (h+r-c)\bar{F}_D(q) + h + (\theta+c)w_{ex}(q_c^*)$. This $g_{\mathbf{x}}(q)$ is an increasing function with respect to q. Now, let $q_h^*(\mathbf{x_1})$ and $q_h^*(\mathbf{x_2})$ be the unique, finite solutions to $g_{\mathbf{x_1}}(q) = 0$ and $g_{\mathbf{x_2}}(q) = 0$, respectively. Since $\mathbf{x_1} \ge \mathbf{x_2}$ component-wise $\implies A_m(\mathbf{x_1}) \ge_{FSD} A_m(\mathbf{x_2})$ (Lemma 1) $\iff F_{A_m(\mathbf{x_1})}(q) \le F_{A_m(\mathbf{x_2})}(q)$, $\forall q \in \mathbb{R}$ (Definition 1), it follows that $g_{\mathbf{x_1}}(q) \le g_{\mathbf{x_2}}(q), \forall q \in \mathbb{R}$. In particular, at $q = q_h^*(\mathbf{x_2})$, we obtain $g_{\mathbf{x_1}}(q_h^*(\mathbf{x_2})) \le g_{\mathbf{x_2}}(q_h^*(\mathbf{x_2})) = 0$, which implies $q_h^*(\mathbf{x_1}) \ge q_h^*(\mathbf{x_2})$. Furthermore, $q_h^*(\mathbf{x_1}) \ge q_h^*(\mathbf{x_2})$ implies $\bar{F}_D(q_h^*(\mathbf{x_1})) \le \bar{F}_D(q_h^*(\mathbf{x_2}))$ because $\bar{F}_D(q)$ is a decreasing function of q. Combining this result with Eq. (25), we obtain $F_{A_m(\mathbf{x_1})}(q_h^*(\mathbf{x_1})) \le F_{A_m(\mathbf{x_2})}(q_h^*(\mathbf{x_2}))$. □

Let $q^\dagger = q_h^*(\mathbf{0})$, and $q^\ddagger = \lim_{v\to\infty} q_h^*(\mathbf{y})$, where v is the smallest component in the initial inventory vector $\mathbf{y}$. Also, let $A_m = A_m(\mathbf{0}) = \sum_{j=1}^m D_j$, which is the sum of m i.i.d. demand random variables. Then, $q_h^*(\mathbf{x})$ is bounded as follows:

Lemma 6. *$q^\dagger \le q_h^*(\mathbf{x}) \le q^\ddagger$ and $F_{A_m}(q^\dagger) \ge F_{A_m(\mathbf{x})}(q_h^*(\mathbf{x}))$, $\forall \mathbf{x} \in \Omega$.*

Note that the lower bound $q^\dagger$ is always finite and tight because $\mathbf{x} = \mathbf{0}$ is recurrent, while the upper bound $q^\ddagger$ is not necessarily finite nor tight because an arbitrarily large initial inventory may not be recurrent.

Proof of Lemma 6. From Lemma 5, we have $q_h^*(\mathbf{0}) \le q_h^*(\mathbf{x}) \le q_h^*(\mathbf{y})$ and $F_{A_m(\mathbf{0})}(q_h^*(\mathbf{0})) \ge F_{A_m(\mathbf{x})}(q_h^*(\mathbf{x})), \forall \mathbf{y} \ge \mathbf{x} (\in \Omega)$ component-wise. We obtain the result by taking the limit of a large initial inventory $\mathbf{y}$. □

Proof of Proposition 4. We split the proof in three parts:

(First part: $F_{A_m}(q^\dagger) \to 0 \implies |q^\ddagger - q^\dagger| \to 0$) Using Lemma 6, $F_{A_m}(q^\dagger) \to 0 \implies F_{A_m(\mathbf{x})}(q_h^*(\mathbf{x})) \to 0, \forall \mathbf{x} \in \Omega$. Since $F_{A_m(\mathbf{x})}(q_h^*(\mathbf{x}))$ converges to the same value (0) for any initial inventory vector $\mathbf{x}$, using Corollary 3, we can conclude $|q^\ddagger - q^\dagger| \to 0$.

(Second part: $|q^\ddagger - q^\dagger| \to 0 \implies q_h^*(\mathbf{x}) \to q_c^*, \forall \mathbf{x} \in \Omega_r$) This part is trivial because $\Omega_r \subseteq \Omega$.

(Third part: $q_h^*(\mathbf{x}) \to q_c^*, \forall \mathbf{x} \in \Omega_r \implies F_{A_m}(q^\dagger) \to 0$) Consider two initial inventory vectors: $\mathbf{x_1} = \mathbf{0}$ and $\mathbf{x_2} = (q^\dagger, 0, \ldots, 0)$; $\mathbf{x_2}$ represents $q^\dagger (= q_h^*(\mathbf{0}) = q_h^*(\mathbf{x_1}))$ units of initial inventory with remaining lifetime of $m-1$ periods. Note that $\mathbf{x_1} \ne \mathbf{x_2}$ and both $\mathbf{x_1}, \mathbf{x_2} \in \Omega_r$ because we assume that D can take 0 and any large amount. Note also that from Proposition 1, we know $F_{A_m(\mathbf{x_2})}(q^\dagger) = 0$. (This is intuitively obvious: $A_m(\mathbf{x_2})$ is the total outflow (through demand and wastage) from periods 1 to m (excluding the wastage in period m) when the initial inventory is $\mathbf{x_2}$. Hence, the support of its CDF is bounded below by $q^\dagger$.) Now, suppose $q_h^*(\mathbf{x}) \to q_c^*, \forall \mathbf{x} \in \Omega_r$, then $q_h^*(\mathbf{x_2}) \to q_h^*(\mathbf{x_1}) = q^\dagger$. Therefore, using Corollary 3 and replacing $q_h^*(\mathbf{x_2})$ with $q^\dagger$, we obtain $F_{A_m}(q^\dagger) = F_{A_m(\mathbf{x_1})}(q_h^*(\mathbf{x_1})) \to F_{A_m(\mathbf{x_2})}(q_h^*(\mathbf{x_2})) \to F_{A_m(\mathbf{x_2})}(q^\dagger) = 0$. □

To prove Corollaries 1 and 2, we define a critical ratio $\gamma^* = \frac{r-c}{h+r-c}$, which is used to determine the optimal service level when q_h^* becomes CBS.

Proof of Corollary 1. Using Proposition 4, $F_{A_m}(q^\dagger) \to 0 \implies F_{A_m(\mathbf{x})}(q_h^*(\mathbf{x})) \to 0, \forall \mathbf{x} \in \Omega \implies f_{A_m(\mathbf{x})}(q) \to 0, \forall q \in [0, q_h^*(\mathbf{x})], \forall \mathbf{x} \in \Omega \implies n_w(q_h^*) = E_{A_m(\mathbf{X})}[q_h^* - A_m(\mathbf{X})]^+ \to 0$, where $n_w(q)$ is defined in Eq. (14). Furthermore, Proposition 4 implies $F_{A_m}(q^\dagger) \to 0 \implies f_{A_m(\mathbf{x})}(q) \to 0, \forall q \in [0, q_c^*], \forall \mathbf{x} \in \Omega_r \implies w_{ex}(q_c^*) \to 0$, where $w_{ex}(q_c)$ is defined in Eq. (21). Lastly, in the limit of $F_{A_m(\mathbf{x})}(q_h^*(\mathbf{x})) \to 0$ and $w_{ex}(q_c^*) \to 0$ in Eq. (25), we obtain a revised optimality condition $(h+r-c)\bar{F}_D(q) = h$, which provides a solution $q_h^* = q_c^* = F_D^{-1}\left(\frac{r-c}{h+r-c}\right) = F_D^{-1}(\gamma^*)$. □

Proof of Corollary 2. Corollary 1 implies that the condition to make q_h^* approach CBS is $F_{A_m}(F_D^{-1}(\gamma^*)) \to 0$. Using a predetermined threshold $\alpha > 0$, we can rewrite this condition as $F_{A_m}(F_D^{-1}(\gamma^*)) \leq \alpha$. □

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